Homework 2

Solutions

For multiple Solutions we will use the following Theorem:

Theorem 1 (Compliment Rule). *Let A be an event in sample space S. Then $P(A^C) = 1 - P(A)$*

Proof.

$$\begin{aligned}
P(S) &= 1 \leftarrow \text{Second Rule of Probability} \\
P(S) &= P(A \cup A^C) \leftarrow \text{ Definition of Compliment} \\
&= P(A) + P(A^C) \leftarrow 4^{th} \text{ rule of probability} \\
\Rightarrow 1 &= P(A) + P(A^C) \leftarrow \text{Combiniting the two equalities above} \\
\Rightarrow 1 - P(A) &= P(A^C)
\end{aligned}$$

□

1. Problem 2.94 from the book (p 60)
   Solution:
   Define the events A: device A detects smoke B: device B detects smoke
   a. $P(A \cup B) = .95 + .90 - .88 = 0.97$.
   b. P(smoke is undetected) = 1 - $P(A \cup B) = 1 - 0.97 = 0.03$.

2. Let A and B be events of a sample space S. Prove that
   $$P(A \cap B) \geq 1 - P(A^C) - P(B^C)$$
   Solution:
   NOTE: there are multiple proofs of this inequality, only one is presented here.
   We know from the general additive rule we have:
   $$\begin{aligned}
   P(A \cup B) &= P(A) + P(B) - P(A \cap B) \leftarrow \text{General Additive Rule} \\
   \Rightarrow P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\
   &= (1 - P(A^C)) + (1 - P(B^C)) - P(A \cup B) \leftarrow \text{Compliment rule from above} \\
   &= 1 - P(A^C) - P(B^C) + (1 - P(A \cup B))
   \end{aligned}$$

We know that $1 \geq P(A \cup B) \geq 0$, which means that $1 - P(A \cup B) \geq 0$. Therefore we have

$$\begin{aligned} P(A \cap B) &= 1 - P(A^C) - P(B^C) + (1 - P(A \cup B)) \\ &\geq 1 - P(A^C) - P(B^C) \end{aligned}$$

3. Problem 2.116 from the book (p 69)
Solution:
LEt A, B, and C be the events that line I, line II, and line III fail, respectively, and let U be the event that all three lines fail (this implies that $U = A \cap B \cap C$). Therefore, we see that U^C is the event in which not all three lines fail. So,

$$\begin{aligned} P(U^C) &= 1 - P(U) \leftarrow \text{Compliment Rule} \\ &= 1 - P(A \cap B \cap C) \leftarrow \text{Definition of } U \\ &= 1 - (P(A)P(B)P(C)) \leftarrow \text{Since } A, B, \text{ and } C \text{ are independent events} \\ &= 1 - (.01)^3 \\ &= 1 - .000001 \\ &= .999999 \end{aligned}$$

4. Let A and B be events of a sample space S. Prove that if $P(A|B) = P(A|B^C)$, then A and B are independent events
Solution:
In order to demonstrate independence, we must show that $P(A \cap B) = P(A)P(B)$

$$\begin{aligned} P(A|B) &= P(A|B^C) \leftarrow \text{By assumption} \\ \Rightarrow \frac{P(A \cap B)}{P(B)} &= \frac{P(A \cap B^C)}{P(B^C)} \leftarrow \text{By definition} \\ \Rightarrow P(A \cap B)P(B^C) &= P(A \cap B^C)P(B) \\ \Rightarrow P(A \cap B)(1 - P(B)) &= P(A \cap B^C)P(B) \leftarrow \text{Compliment Rule} \\ \Rightarrow P(A \cap B) - P(A \cap B)P(B) &= P(A \cap B^C)P(B) \\ \Rightarrow P(A \cap B) &= P(A \cap B^C)P(B) + P(A \cap B)P(B) \\ &= [P(A \cap B^C) + P(A \cap B)]P(B) \\ &= P(A)P(B) \end{aligned}$$

Thus, A and B must be independent.

5. Suppose that we draw a playing card from a standard 52-card deck. Is the event that we draw a face card (i.e. Jack, Queen, King) independent from the event that we draw a heart?

Solution:
Let ☼ be the event that we draw a face card and let ☺ be the event that we draw a heart

$$
\begin{aligned}
P(\text{☼}) &= \frac{\#\text{ of ways to draw a face card}}{\#\text{ of ways to draw a card in general}} \\
&= \frac{3\times 4}{52} \\
&= \frac{12}{52} = \frac{3}{13}
\end{aligned}
$$

$$
\begin{aligned}
P(\text{☺}) &= \frac{\#\text{ of ways to draw a heart}}{\#\text{ of ways to draw a card in general}} \\
&= \frac{13}{52} = \frac{1}{4}
\end{aligned}
$$

$$
\begin{aligned}
P(\text{☼} \cap \text{☺}) &= \frac{\#\text{ of ways to draw a heart suit face card}}{\#\text{ of ways to draw a card in general}} \\
&= \frac{3\times 1}{52} = \frac{3}{52} \\
P(\text{☼})P(\text{☺}) &= \left(\frac{3}{13}\right)\left(\frac{1}{4}\right) \\
&= \frac{3}{13\times 4} = \frac{3}{52}
\end{aligned}
$$

$$\Rightarrow P(\text{☼})P(\text{☺}) = P(\text{☼} \cap \text{☺})$$

Therefore we see that by definition, the event that we draw a face card and the event that we draw a heart are independent events