1 Motivation & Definition

- We will begin to examine various Continuous Random variables, and we will start with one of the Simplest, the continuous Uniform Random Variable
- Suppose that you are waiting for the bus and the schedule says that the bus is to arrive at noon, but admits that the bus may be anywhere between 5 minutes early and 5 minutes late.
- Suppose that the probability density was evenly spread out over the whole range of possible times that the bus could arrive. What would the distribution of the arrival time of the bus look like?
- First, lets Define X to be the arrival time, in terms of minutes from noon.
- This means that the support of X is the interval $(-5, 5)$
- When we say that the probability density is evenly spread out over the range of possible values, that means that the PDF is the sam value, no matter what possible value is put into the function (i.e. $f_X(x) = c \forall x \in S$, the support of X where c is a constant, and $f_X(x) = 0 \forall x \notin S$)
- So, what is this constant value?
- Well, we know that in order to be a valid PDF $\int_{-\infty}^{\infty} f_X(x) \mathrm{d}x = 1$
- But the support is only $(-5, 5)$ (which correspond to $11:55$ and $12:05$). So the PDF is just 0 for all values outside of the support
- Because the integral of 0 is just 0 over the intervals of the real line that aren't the support of X, we that

$$\int_{-\infty}^{\infty} f_X(x) \mathrm{d}x = \int_{-5}^{5} f_X(x) \mathrm{d}x = 1$$

- Furthermore, we know that the PDF is a constant for all values in S, so we can substitute c in for $f_X(x)$ in th integral and then solve for c

$$\begin{aligned} \int_{-5}^{5} f_X(x) \mathrm{d}x &= \int_{-5}^{5} c \mathrm{d}x \\ &= cx|_{x=-5}^{5} \\ &= (5c) - (-5c) \\ &= 10c \end{aligned}$$

Since

$$\begin{aligned}
\int_{-\infty}^{\infty} f_X(x)\mathrm{d}x &= 1 \\
\text{We see that} & \\
10c &= 1 \\
\Rightarrow c &= \frac{1}{10}
\end{aligned}$$

- Now suppose that we generalize this problem. Suppose that the bus may arrive any where from a minutes from noon to b minutes from noon, where $a < b$
- This means that our support would then be (a, b)
- Again, we are assuming the probability density is evenly spread out over the possible times that teh bus mght show up (i.e. evenly spread over the support)
- So, we will assume that

$$f_X(x) = \begin{cases} c & \text{if } a < x < b \\ 0 & \text{else} \end{cases}$$

- We will again solve for the value of c

$$\begin{aligned}
\int_{-\infty}^{\infty} f_X(x)\mathrm{d}x &= \int_a^b f_X(x)\mathrm{d}x \\
&= \int_a^b c\mathrm{d}x \\
&= cx|_{x=a}^{b} \\
&= (bc) - (ac) \\
&= (b-a)c \\
\text{Since} & \\
\int_{-\infty}^{\infty} f_X(x)\mathrm{d}x &= 1 \\
\text{We see that} & \\
(b-a)c &= 1 \\
\Rightarrow c &= \frac{1}{b-a}
\end{aligned}$$

- When a Random Variable X has the support (a, b) where a and b are real valued constants where $a < b$ and the pdf

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{else} \end{cases}$$

  We say that X has a *Uniform* Distribution. (Symobllically, represented as $X \sim U(a, b)$)

2 Verify, Mean, Variance, & MGF

Let $X \sim U(a,b)$

1. Verify

   a) $f_X(x) \geq 0 \forall x \in S$

      Proof. The support of X is (a,b) and $a < b$. This means that $\frac{1}{b-a} > 0$ and since $f_X(x) = \frac{1}{b-a} \forall x \in (a,b) = S$ we see that $f_X(x) \geq 0 \forall x \in S$ $\square$

   b) $\int_{-\infty}^{\infty} f_X(x) \mathrm{d}x = 1$

      Proof.

$$\begin{aligned}
\int_{-\infty}^{\infty} f_X(x)\mathrm{d}x &= \int_a^b f_X(x)\mathrm{d}x \leftarrow \text{Since } f_X(x) = 0 \forall x \notin (a,b) \\
&= \int_a^b \frac{1}{b-a}\mathrm{d}x \\
&= \frac{1}{b-a} c|_{x=a}^{b} \\
&= \frac{b}{b-a} - \frac{a}{b-a} \\
&= \frac{b-a}{b-a} \\
&= 1
\end{aligned}$$

$\square$

2. Mean

$$E[X] = \frac{1}{2}(a+b)$$

   Proof.

$$\begin{aligned}
E[X] &= \int_a^b x f_X(x)\mathrm{d}x \\
&= \int_a^b x \frac{1}{b-a}\mathrm{d}x \\
&= \frac{1}{2}\frac{x^2}{b-a}|_{x=a}^{b} \\
&= \frac{1}{2}[\frac{b^2}{b-a} - \frac{a^2}{b-a}]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{b^2 - a^2}{2(b-a)} \\
&= \frac{(b-a)(b+a)}{2(b-a)} \\
&= \frac{1}{2}(b+a)
\end{aligned}
$$

□

3. Variance
$$V[X] = \frac{(b-a)^2}{12}$$

Proof. Proof left as Extra Credit □

4. MGF
$$M_X(t) = \begin{cases} \frac{e^{bt}-e^{at}}{t(b-a)} & t \neq 0 \\ 1 & t = 0 \end{cases}$$

Proof. Proof left as Homework □