Homework 6

SOLUTIONS!

1. Problem 4.25 from the book (p 172)
   Solution:
   First we must determine the PDF of Y

$$
\begin{aligned}
f_Y(y) &= \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y) \\
&= \begin{cases} \frac{\mathrm{d}}{\mathrm{d}y} 0, & y \leq 0 \\ \frac{\mathrm{d}}{\mathrm{d}y} \frac{y}{8}, & 0 < y < 0 \\ \frac{\mathrm{d}}{\mathrm{d}y} \frac{y^2}{16}, & 2 \leq y < 4 \\ \frac{\mathrm{d}}{\mathrm{d}y} 1, & y \geq 4 \end{cases} \\
&= \begin{cases} 0, & y \leq 0 \\ \frac{1}{8}, & 0 < y < 0 \\ \frac{y}{8}, & 2 \leq y < 4 \\ 01, & y \geq 4 \end{cases}
\end{aligned}
$$

   So, That means that the Expectation is

$$
\begin{aligned}
E[Y] &= \int_{-\infty}^{\infty} y f_Y(y) \mathrm{d}y \\
&= \int_0^4 y f_Y(y) \mathrm{d}y \leftarrow \text{Since } f_Y(y) = 0 \text{ when } y \leq 0 \text{ or } y \geq 4 \\
&= \int_0^2 y \frac{1}{8} \mathrm{d}y + \int_2^4 y \frac{y}{8} \mathrm{d}y \\
&= \frac{y^2}{16} \Big|_{y=0}^2 + \frac{y^3}{24} \Big|_{y=2}^4 \\
&= \frac{1}{4} + \frac{4^3}{24} - \frac{2^3}{24} \\
&= \frac{3}{12} + \frac{2(4^2)}{12} - \frac{2^2}{12} \\
&= \frac{31}{12}
\end{aligned}
$$

Thus the Variance is

$$\begin{aligned}
V[Y] &= E[Y^2] - E^2[Y] \\
\text{where} & \\
E[Y^2] &= \int_{-\infty}^{\infty} y^2 f_Y(y)\mathrm{d}y \\
&= \int_0^4 y^2 f_Y(y)\mathrm{d}y \leftarrow \text{Since } f_Y(y) = 0 \text{ when } y \leq 0 \text{ or } y \geq 4 \\
&= \int_0^2 y^2 \frac{1}{8}\mathrm{d}y + \int_2^4 y^2 \frac{y}{8}\mathrm{d}y \\
&= \frac{y^3}{24}|_{y=0}^2 + \frac{y^4}{32}|_{y=2}^4 \\
&= \frac{2^3}{24} + \frac{4^4}{32} - \frac{2^4}{32} \\
&= \frac{2}{6} + \frac{3(4^2)}{6} - \frac{3}{6} \\
&= \frac{47}{6} \\
\Rightarrow V[Y] &= \frac{47}{6} - (\frac{31}{12})^2 \\
&= \frac{47(24)}{12^2} - \frac{31^2}{12^2} \\
&= \frac{167}{12^2}
\end{aligned}$$

2. Let $X \sim U(a,b)$. Derive the CDF of X
   Solution:

$$\begin{aligned}
F_X &= \int_{-\infty}^{x} f_X(y)\mathrm{d}y \\
&= \begin{cases} \int_{-\infty}^{x} f_X(y)\mathrm{d}y, & x \leq a \\ \int_{-\infty}^{a} f_X(y)\mathrm{d}y + \int_a^x f_X(y)\mathrm{d}y, & a < x < b \\ \int_{-\infty}^{a} f_X(y)\mathrm{d}y + \int_a^b f_X(y)\mathrm{d}y + \int_b^x f_X(y)\mathrm{d}y, & b \leq x \end{cases} \\
&= \begin{cases} \int_{-\infty}^{x} 0\mathrm{d}y, & x \leq a \\ \int_{-\infty}^{a} 0\mathrm{d}y + \int_a^x \frac{1}{b-a}\mathrm{d}y, & a < x < b \\ \int_{-\infty}^{a} 0\mathrm{d}y + \int_a^b \frac{1}{b-a}\mathrm{d}y + \int_b^x 0\mathrm{d}y, & b \leq x \end{cases} \\
& \uparrow \text{Because } f_X(y) = 0 \text{ when } y \leq a \text{ or } y \geq b \\
&= \begin{cases} 0, & x \leq a \\ \frac{y}{b-a}|_{y=a}^x, & a < x < b \\ \frac{y}{b-a}|_{y=a}^b, & b \leq x \end{cases}
\end{aligned}$$

$$= \begin{cases} 0, & x \le a \\ \frac{x-a}{b-a}, & a < x < b \\ 1, & b \le x \end{cases}$$

3. Let $X \sim U(a,b)$. Derive $\phi_{.5}$, the $\frac{1}{2}$ Quantile or *Median* of the distribution of X
   Solution:
   By Definition, $\phi_{.5}$ is the (smallest) value such that $P(X < \phi_{.5}) = F_X(\phi_{.5}) = .5$

$$\begin{aligned} F_X(\phi_{.5}) &= .5 \\ \Rightarrow \frac{\phi_{.5} - a}{b-a} &= .5 \leftarrow \text{Since } a < \phi_{.5} < b \\ \Rightarrow \phi_{.5} &= .5(.b-a) + a \\ &= \frac{a+b}{2} \end{aligned}$$

4. Let $X \sim U(a,b)$ Show that the MGF is

$$M_X(t) = \begin{cases} \frac{e^{tb}-e^{ta}}{t(b-a)} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$$

   Note: The MGF is written this way because $\frac{e^{tb}-e^{ta}}{t(b-a)}$ is of indeterminate form at $t=0$ (it comes out to be $\frac{0}{0}$). So, we must show that This indeterminate form really works out to be 1 at $t=0$. We Can do this by showing that $\lim_{t\to 0} M_X(t)$ exists. If the limit exists, then $M_X(t)|_{t=0} = \lim_{t\to 0} M_X(t)$
   Solution:

$$\begin{aligned} M_X(t) &= E[e^{Xt}] \\ &= \int_{-\infty}^{\infty} e^{xt} f_X(x)\mathrm{d}x \\ &= \int_a^b e^{xt} f_X(x)\mathrm{d}x \\ &= \int_a^b e^{xt} \frac{1}{b-a}\mathrm{d}x \\ &= \int_a^b e^{xt} \frac{1}{b-a}\mathrm{d}x \\ &= \frac{e^{xt}}{t(b-a)}\Big|_{x=a}^{b} \leftarrow \text{Only if } t \neq 0 \\ &= \frac{e^{bt}-e^{at}}{t(b-a)} \end{aligned}$$

So, what happens when $t = 0$? We need the MGF to be defined at $t = 0$ and we see that

$$
\begin{aligned}
M_X(t)|_{t=0} &= E[e^{Xt}]_{t=0} \\
&= \frac{e^{bt} - e^{at}}{t(b-a)}|_{t=0} \\
&= \frac{0}{0}
\end{aligned}
$$

This is an indeterminate form, which means that $M_X(t)|_{t=0} = \lim_{t\to 0} M_X(t)$, if the limit exists

$$
\begin{aligned}
\lim_{t\to 0} M_X(t) &= \lim_{t\to 0} \frac{e^{bt} - e^{at}}{t(b-a)} \\
&= \lim_{t\to 0} \frac{be^{bt} - ae^{at}}{(b-a)} \leftarrow \text{By L'Hospital's Rule} \\
&= \frac{b-a}{b-a} \\
&= 1
\end{aligned}
$$

Thus, we conclude that

$$
M_X(t) = \begin{cases} \frac{e^{tb} - e^{ta}}{b-a} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}
$$