1 Set Theory Definitions

- A *set* is simply a collection of objects:
  - The set of counting numbers
  - The set of students in this classroom
  - The set of professors in the statistics department
- Typical notation: $A = \{\dots\}$
- Examples:
  - $N = \{0, 1, 2, \dots\}$ (the set of natural numbers)
  - $Z = \{\dots - 2, -1, 0, 1, 2, \dots\}$ (the set of integers)
  - $A = \{1, 2, 3\}$
  - $B = \{4, 5, 6\}$
  - $C = \{7, 8, 9\}$
  - $D = \{1, 2, 4\}$
- We say that
  - 0 is an *element* of set Z and set N (written $0 \in Z$)
  - 1 is an *element* of set Z, set N, and set A (written $1 \in Z$, $1 \in N$, and $1 \in A$
  - 1 is NOT an *element* of set C (written $1 \notin C$)
- Two operations for sets
  - Set *intersection*, denoted $\cap$, i.e. $A \cap B$
    * The intersection of two sets is the set of all elements in both sets
    * i.e. In more rigorous terms, a set C is the intersection of sets A and B (i.e. $C = A \cap B$) if and only if the following implications are true:
      1. $a \in C \Rightarrow a \in A$ and $a \in B$ (If a is an element of C, then a is an element of both set A and set B)
      2. a such that $a \in A$ and $a \in B \Rightarrow a \in C$ (If a is an element of both set A and set B, then a is an element of C)
  - Set *union*, denoted $\cup$, i.e. $A \cup B$
    * The union of two sets is the set of all elements in either set

- i.e. In more rigorous terms, a set C is the union of sets A and B (i.e. $C = A \cup B$) if and only if the following implications are true:
  1. $a \in C \Rightarrow a \in A$ or $a \in B$ (If a is an element of C, then a is an element of set A or set B)
  2. a such that $a \in A$ or $a \in B \Rightarrow a \in C$ (If a is an element of set A or set B, then a is an element of C)

- Examples:
  - $A \cup B = \{1, 2, 3, 4, 5, 6\}$
  - $A \cup C = \{1, 2, 3, 7, 8, 9\}$
  - $A \cap D = \{1, 2\}$
  - $B \cap D = \{4\}$
  - $A \cup N = ?(N)$
  - $A \cap N = ?(A)$
  - $A \cap B = ?(\varnothing, \text{the empty set})$

- Subsets
  - A set (A) is a *subset* of another set (B) is a set whose elements are all elements of another set (denoted $A \subset B$)
  - Formally, we say that A is a subset of B if and only if $a \in A \Rightarrow a \in B$
  - Here A would be the subset and B would be the *superset*
  - Consider sets A, N, and Z from above
    - We see the every element in N is in Z, thus $N \subset Z$
    - Similarly, every element in A is an element in N, thus $A \subset N$
    - Is A a subset of Z? (Yes)

- Disjoint
  - If sets $A_1, A_2, ..., A_n$ is a sequence of n sets for some whole number n are such that $A_1 \cap A_2 \cap ... \cap A_n = \varnothing$, then sets $A_1, ..., A_n$ are said to be *disjoint*
  - If sets $A_1, A_2, ..., A_n$ are subsets of set S and $A_1, A_2, ..., A_n$ are disjoint and $A_1 \cup A_2 \cup ... \cup A_n = S$ then sets $A_1, A_2, ..., A_n$ form a *partition*

- Compliment
  - The *compliment* of a subset (A) with respect to a supert set (S) is the set of every element in the superset that is not an element of the subset
  - We denoted the compliment of a set A, with the notation A^C or $\overline{A}$
  - Formally, we say that B is the compliment of A (i.e. $B = A^C$) with respect to a super set S if and only if
    1. $A \cap B = \varnothing$ and,
    2. $A \cup B = S$
  - Examples

* Consider set $A = \{1, 2, 3\}$
* Let set $J = \{1\}$ and set $K = \{1, 2\}$
* We see that $J \subset A$ and $K \subset A$
* Then $J^C = \{2, 3\}$
* And $K^C = \{3\}$

2 Exercises:

1. LET $E = \{a, b, c\}$, $F = \{c, d, e\}$, $G = \{1, 2, 3\}$, $H = \{a, 2\}$, $I = \{c\}$
   (a) $E \cup F = ?(\{a, b, c, d, e\})$
   (b) $F \cap G = ?(\varnothing)$
   (c) $(I \cap E) \cap F = ?(I \cap F = I)$
   (d) $(E \cup H) \cap G = ?(\{a, b, c, 2\} \cap G = \{2\})$
2. Let $U = \{\text{natural numbers}\}$, $A = \{2, 4, 6, 8, 10\}$, $B = \{1, 3, 6, 7, 8\}$. State whether each of the following is true or false:
   (a) $2 \in A$ (T)
   (b) $11 \in B$ (F)
   (c) $4 \notin B$ (T)
   (d) $A \in U$ (F)
   (e) $A = \{\text{even numbers}\}$ (F)
3. Let $A = \{g, 4, +\sqrt{2}, 2/3, \$ - 2.5, \&, -5, v, 33, +\sqrt{9}, \pi\}$. Using the $\{\dots\}$ set notation, write the sets of:
   (a) Natural numbers in A $(\{4, 33, +\sqrt{9}\})$
   (b) Integers in A $(\{4, -5, 33, +\sqrt{9}\})$
   (c) Irrational numbers in A $(+\sqrt{2}, \pi)$
   (d) Non alpha-numeric symbols that do not represent a numeric value in A $(\$, \&)$
4. True or false?
   (a) $\varnothing = \{0\}$ (F)
   (b) $x \in \{x\}$ (T)
   (c) $\varnothing = \{\varnothing\}$ (F)
   (d) $\varnothing \in \{\varnothing\}$ (T)

3 Set Theory Laws

For all of the following Laws we will let A, B, and C represent different sets.

* Distriubtive Laws
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof. to show that two sets are equal, we must show that each set is a subset of the other, so we will proceed in two steps:

1. $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$
   To show that one set is a subset of another, we must show that if an element is in the subset, then it must be in the other (see the section on subsets).
   Let ☺ $\in A \cup (B \cap C)$

$$\begin{array}{rcl} \Rightarrow ☺ \in A & \text{or} & ☺ \in (B \cap C) \\ \text{If } ☺ \in A \text{ then:} & & \\ ☺ \in (A \cup B) & \text{and} & ☺ \in (A \cup C) \leftarrow \text{Since } ☺ \in A \\ \Rightarrow ☺ & \in & (A \cup B) \cap (A \cup C) \end{array}$$

   If ☺ $\in (B \cap C)$ then:

$$\begin{array}{rcl} ☺ \in B & \text{and} & ☺ \in C \leftarrow \text{From the definintion of Set intersection} \\ ☺ & \in & (A \cup B) \leftarrow \text{Since } ☺ \in B \\ ☺ & \in & (A \cup C) \leftarrow \text{Since } ☺ \in C \\ \Rightarrow ☺ & \in & (A \cup B) \cap (A \cup C) \end{array}$$

   Now, we must show that the subsetting goes the other way as well.

2. $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$
   To show that one set is a subset of another, we must show that if an element is in the subset, then it must be in the other.
   Let ☺ $\in (A \cup B) \cap (A \cup C)$

$$\begin{array}{rcl} \Rightarrow ☺ \in (A \cup B) & \text{and} & ☺ \in (A \cup C) \\ \text{If } ☺ \in A \text{ then:} & & \\ \Rightarrow ☺ & \in & A \cup (B \cap C) \leftarrow \text{Since } ☺ \in A \\ \text{If } ☺ \notin A \text{ then:} & & \\ ☺ \in B & \text{and} & ☺ \in C \leftarrow \text{since } ☺ \in (A \cup B) \text{ and } ☺ \in (A \cup C) \\ ☺ & \in & (B \cap C) \leftarrow \text{Since } ☺ \in B \text{ and } ☺ \in C \\ ☺ & \in & A \cup (B \cap C) \end{array}$$

Thus, we have shown that both sets are subsets of the other. Therefore the two sets must be the same. □

– $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof. To show that two sets are equal, we must show that each set is a subset of the other, so we will proceed in two steps:

1. $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$
   To show that one set is a subset of another, we must show that if an element is in the subset, then it must be in the other (see the section on subsets).

Let $☺ \in A \cap (B \cup C)$

$$\begin{array}{rcl} \Rightarrow ☺ \in A & \text{and} & ☺ \in (B \cup C) \\ \Rightarrow ☺ \in B & \text{or} & ☺ \in C \leftarrow \text{Since } ☺ \in (B \cup C) \end{array}$$

If $☺ \in B$ then:

$$\begin{array}{rcl} \Rightarrow ☺ & \in & A \cap B \\ \Rightarrow ☺ & \in & (A \cap B) \cup (A \cap C) \end{array}$$

If $☺ \in C$ then:

$$\begin{array}{rcl} \Rightarrow ☺ & \in & A \cap C \\ \Rightarrow ☺ & \in & (A \cap B) \cup (A \cap C) \end{array}$$

Now, we must show that the subsetting goes the other way as well.

2. $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$
To show that one set is a subset of another, we must show that if an element is in the subset, then it must be in the other.
Let $☺ \in (A \cap B) \cup (A \cap C)$

$$\Rightarrow ☺ \in (A \cap B) \quad \text{or} \quad ☺ \in (A \cap C)$$

If $☺ \in (A \cap B)$ then:

$$\begin{array}{rcl} ☺ \in A & \text{and} & ☺ \in B \leftarrow \text{Since } ☺ \in (A \cap B) \\ \Rightarrow ☺ & \in & (B \cup C) \leftarrow \text{Since } ☺ \in B \\ \Rightarrow ☺ & \in & A \cap (B \cup C) \leftarrow \text{Since } ☺ \in A \text{ and } ☺ \in (B \cup C) \end{array}$$

If $☺ \in (A \cap C)$ then:

$$\begin{array}{rcl} ☺ \in A & \text{and} & ☺ \in C \leftarrow \text{Since } ☺ \in (A \cap C) \\ \Rightarrow ☺ & \in & (B \cup C) \leftarrow \text{Since } ☺ \in C \\ \Rightarrow ☺ & \in & A \cap (B \cup C) \leftarrow \text{Since } ☺ \in A \text{ and } ☺ \in (B \cup C) \end{array}$$

Thus, we have shown that both sets are subsets of the other. Therefore the two sets must be the same. □

- Associative Laws
  - $(A \cup B) \cup C = A \cup (B \cup C)$

Proof. 1. $(A \cup B) \cup C \subset A \cup (B \cup C)$
Let $☼ \in (A \cup B) \cup C$

$$\Rightarrow ☼ \in (A \cup B) \quad \text{or} \quad ☼ \in C$$

If $☼ \in C$ then:

$$\begin{array}{rcl} ☼ & \in & (B \cup C) \\ \Rightarrow ☼ & \in & A \cup (B \cup C) \end{array}$$

If $☼ \in (A \cup B)$ then:

$$☼ \in A \quad \text{or} \quad ☼ \in B$$

If $\star \in A$ then:

$$\star \in A \cup (B \cup C)$$

If $\star \in B$ then:

$$\begin{aligned} \star &\in (B \cup C) \\ \Rightarrow \star &\in A \cup (B \cup C) \end{aligned}$$

2. $A \cup (B \cup C) \subset (A \cup B) \cup C$
   Let $\star \in A \cup (B \cup C)$

$$\Rightarrow \star \in A \quad \text{or} \quad \star \in (B \cup C)$$

If $\star \in A$ then:

$$\begin{aligned} \star &\in (A \cup B) \\ \Rightarrow \star &\in (A \cup B) \cup C \end{aligned}$$

If $\star \in (B \cup C)$ then:

$$\star \in B \quad \text{or} \quad \star \in C$$

If $\star \in C$ then:

$$\star \in (A \cup B) \cup C$$

If $\star \in B$ then:

$$\begin{aligned} \star &\in (A \cup B) \\ \Rightarrow \star &\in (A \cup B) \cup C \end{aligned}$$

Thus, we have shown that both sets are subsets of the other. Therefore the two sets must be the same. $\square$

– $(A \cap B) \cap C = A \cap (B \cap C)$

Proof. 1. $(A \cap B) \cap C \subset A \cap (B \cap C)$
   Let $\star \in (A \cap B) \cap C$

$$\begin{aligned} \Rightarrow \star \in (A \cap B) &\quad \text{and} \quad \star \in C \\ \Rightarrow \star \in A, \star \in B &\quad \text{and} \quad \star \in C \leftarrow \text{Since } \star \in (A \cap B) \\ \Rightarrow \star &\quad \in \quad (B \cap C) \\ \Rightarrow \star &\quad \in \quad A \cap (B \cap C) \end{aligned}$$

2. $A \cap (B \cap C) \subset (A \cap B) \cap C$
   Let $\star \in A \cap (B \cap C)$

$$\begin{aligned} \Rightarrow \star \in A &\quad \text{and} \quad \star \in (B \cap C) \\ \Rightarrow \star \in A, \star \in B &\quad \text{and} \quad \star \in C \leftarrow \text{Since } \star \in (B \cap C) \\ \Rightarrow \star &\quad \in \quad (A \cap B) \\ \Rightarrow \star &\quad \in \quad (A \cap B) \cap C \end{aligned}$$

Thus, we have shown that both sets are subsets of the other. Therefore the two sets must be the same. $\square$

- Commutative Laws
  - $A \cup B = B \cup A$

    Proof. 1. $A \cup B \subset B \cup A$
    Let $\Bell \in A \cup B$

    $$\begin{aligned} &\Rightarrow \Bell \in A \quad \text{or} \quad \Bell \in B \\ \text{If } \Bell \in A \text{ then :}& \\ \Bell &\in B \cup A \\ \text{If } \Bell \in B \text{ then :}& \\ \Bell &\in B \cup A \end{aligned}$$

    2. $B \cup A \subset A \cup B$
    Let $\Bell \in B \cup A$

    $$\begin{aligned} &\Rightarrow \Bell \in B \quad \text{or} \quad \Bell \in A \\ \text{If } \Bell \in B \text{ then :}& \\ \Bell &\in B \cup A \\ \text{If } \Bell \in A \text{ then :}& \\ \Bell &\in B \cup A \end{aligned}$$

    $\square$

  - $A \cap B = B \cap A$

    Proof. 1. $A \cap B \subset B \cap A$
    Let $\Bell \in A \cap B$

    $$\begin{aligned} &\Rightarrow \Bell \in A \quad \text{and} \quad \Bell \in B \\ &\Rightarrow \Bell \in B \quad \text{and} \quad \Bell \in A \\ &\Rightarrow \Bell \in B \cap A \end{aligned}$$

    2. $B \cap A \subset A \cap B$
    Let $\Bell \in B \cap A$

    $$\begin{aligned} &\Rightarrow \Bell \in B \quad \text{and} \quad \Bell \in A \\ &\Rightarrow \Bell \in A \quad \text{and} \quad \Bell \in B \\ &\Rightarrow \Bell \in A \cap B \end{aligned}$$

    $\square$