1 Revisiting the Coin Flip

- Consider the experiment where we flip a fair coin
  Sample space$\rightarrow \{H, T\}$
- Can we make a random variable for this experiment?
- Let $X = \#$ of heads we observe when we flip a coin
- X is 1 if we get a head, and 0 if we get a tail
- What is the support of X? ($\{0, 1\}$)
- What is the PDF of X?
- $p_X(x)$ will be the function that takes a possible value of X, x, and returns $P(X = x)$
- so

$$\begin{aligned} p_X(0) &= P(X = 0) \\ &= P(T) \\ &= 1/2 \end{aligned}$$

  and

$$\begin{aligned} p_X(0) &= P(X = 0) \\ &= P(T) \\ &= 1/2 \end{aligned}$$

- Now, we can imagine that the coin that we flip is in fact not fair. (in fact, it has been shown that a regular us penny actually favors landing heads up slightly)
- Lets suppose that $P(H) = p$, where p is a probability between 0 and 1
- What would this make P(T)? (1-p)
- So if we return to our function notation, this would be:
  - $p_X(1) = p$
  - $p_X(0) = 1 - p$
- How can we write this function in a formula?
- One formula that we can use to get the desired function is $p_X(x) = p^x(1-p)^{1-x}$

- We say that if a R.V. has the above distribution function, then it is a *Bernoulli* Random variable with a *Bernoulli Distribution*
- We write this as $X \sim Bern(p)$
- This is read as "X has a Bernoulli distribution with probability of success p"

2 Flipping More Coins

- Suppose that we now flip two coins, a quarter and a nickel; both have a probability of landing heads up of p
- What is the sample space? $\rightarrow \{(H,H),(H,T),(T,H),(T,T)\}$
- First lets find the probabilities of each outcome (note the outcome of each coin is independent):
  -
  $$\begin{aligned} P((H,H)) &= P(\text{Quarter lands heads} \cap \text{Nickel lands heads}) \\ &= P(\text{Quarter lands heads})P(\text{Nickel lands heads}) \\ &= p^2 \end{aligned}$$
  -
  $$\begin{aligned} P((H,T)) &= P(\text{Quarter lands heads} \cap \text{Nickel lands heads}) \\ &= P(\text{Quarter lands heads})P(\text{Nickel lands tails}) \\ &= p(1-p) \end{aligned}$$
  -
  $$\begin{aligned} P((T,H)) &= P(\text{Quarter lands tails} \cap \text{Nickel lands heads}) \\ &= P(\text{Quarter lands tails})P(\text{Nickel lands heads}) \\ &= (1-p)p \end{aligned}$$
  -
  $$\begin{aligned} P((T,T)) &= P(\text{Quarter lands tails} \cap \text{Nickel lands heads}) \\ &= P(\text{Quarter lands tails})P(\text{Nickel lands tails}) \\ &= (1-p)^2 \end{aligned}$$
- Now lets make a random variable for this new experiment

- Again we will let $X = \#$ of heads observed in a single trial of the experiment
- What is the supprt of X? $\rightarrow \{0, 1, 2\}$
- Lets find the formula for $p_X(x)$, the distribution function of X
- We know that:

$$\begin{aligned} p_X(0) &= P(X = 0) \\ &= P((T, T)) \\ &= (1 - p)^2 \end{aligned}$$

$$\begin{aligned} p_X(2) &= P(X = 1) \\ &= P((H, H)) \\ &= p^2 \end{aligned}$$

- but what is $p_X(1)$?
- Remember, $p_X(1) = P(X = 1)$, the probability that we observe 1 head
- Because there are two coins there are $\binom{2}{1} = 2$ different ways that we could have one coin land heads up
- And because the coin flips are independent and both have the same probability of coming up heads, the probability of the Quarter landing heads up with the nickel landing heads down will be the same as the nickel landing heads up and the quarter landing heads down
- We can even think about the probability of the events $X = 0$ and $X = 1$ in a similar way
- In the case of $X = 2$, there are $\binom{2}{2} = 1$ ways to select two heads
- In the case of $X = 0$, there are $\binom{2}{0} = 1$ ways to select zero heads
- This means that we can generalize our probility $P(X = x)$ as

  $\#$ of ways to select x heads $\times$ probability of one particular ordering of x heads

- The $\#$ of ways to select x heads will be $\binom{2}{x}$
- And because all orderings will have the same probability (due to independence and all coins having the same probability of landing heads up), we see that the probability of one particular ordering of x heads will be

$$P(\{H\})^x P(\{T\})^{2-x} = p^x (1 - p)^{2-x}$$

- What this means is that our formula for $p_X(x)$ can be written as:

$$p_X(x) = \binom{2}{x} p^2 (1-p)^{2-x}$$

- If we generalize our random variable to be the # of oberseved heads when flipping n coins instead of just 2, hw would our formula change?
- We can see that since all of the coins would have the same probability of coming up heads and would all still be independent, the probability of observing x heads would still be of the form:

$$\text{\# of ways to select } x \text{ heads} \times \text{probability of one particular ordering of } x \text{ heads}$$

- with n coins however, the # of ways to select x heads would be $\binom{n}{x}$ and the probability of one particular ordering of x heads would be

$$P(\{H\})^x P(\{T\})^{2-x} = p^x (1-p)^{n-x}$$

- Thus our formula for the PDF of this random variable would be

$$p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

- If a random variable has this distribution, it is said to be a *Binomial* random variable
- This is written as $X \sim Bin(n, p)$
- This notation is read as X has a binomial distribution with n attempts and p probability of success

3 Verification, Mean, & Variance

- As we establish new probability distriobutions and their accompanying PDFs, we will aim to do three things:
    1. Verify that the PDF is valid (i.e., show that it follows the two rules for discrete R.V. PDFs)
    2. Establish the meand of distribution
    3. Establish the variance of the distribution

3.1 Bernoulli

- We begin with the Bernoulli distribuion. Let $X \sim Bern(p)$
  1. Verification
     a) $0 \leq p_X(x) \leq 1 \forall x \in S$

        Proof. Let $x \in S$, the support of X ($\{0,1\}$). Then $p_X = p^x(1-p)^{1-x}$. Since $0 < p < 1$ we see that $0 < 1 - p < 1$ as well. Thus $0 < p^x \leq 1$ and $0 < (1-p)^{1-x} \leq 1$. Therefore $0 \leq p_X(x) \leq 1 \forall x \in S$ □

     b) $\sum_{x \in S} p_X(x) = 1$

        Proof.

$$\begin{aligned} \sum_{x \in S} p_X(x) &= \sum_{x=0}^{1} p^x(1-p)^{1-x} \\ &= 1 - p + p \\ &= 1 \end{aligned}$$

□

  2. Mean

$$E[X] = p$$

     Proof.

$$\begin{aligned} E[X] &= \sum_{x \in S} x p_X(x) \\ &= \sum_{x=0}^{1} x p^x(1-p)^{1-x} \\ &= 0(1-p) + 1p \\ &= p \end{aligned}$$

□

  3. Variance

$$V[X] = p(1-p)$$

Proof.

$$\begin{aligned} V[X] &= E[X^2] - E^2[X] \leftarrow \text{Variance theorem} \\ &= \sum_{x \in S} x^2 p_X(x) - p^2 \leftarrow \text{Proven above} \\ &= \sum_{x=0}^{1} x^2 p^x (1-p)^{1-x} - p^2 \\ &= 0(1-p) + 1p - p^2 \\ &= p(1-p) \end{aligned}$$

□

3.2 Binomial

- Now we will do the same with the Binomial Distribution. Let $X \sim Bern(p)$
  1. Verification
     - *Note:* We will be making use of what is call the binomial expansion theorem, which states that:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

  a) $\sum_{x \in S} p_X(x) = 1$

  Proof.

$$\begin{aligned} \sum_{x \in S} p_X(x) &= \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} \\ &= (p + (1-p))^n \leftarrow \text{Binomial Expansion Theorem} \\ &= 1^n \\ &= 1 \end{aligned}$$

□

  b) $0 \le p_X(x) \le 1 \forall x \in S$

  Proof. Let $x \in S$, the support of X ($\{0, 1, ..., n\}$). Then $p_X = \binom{n}{x} p^x (1-p)^{1-x}$. Since $0 < p < 1$ we see that $0 < 1-p < 1$ as well. Thus $0 < p^x \le 1$ and $0 < (1-p)^{1-x} \le 1$. Therefore $0 \le \binom{n}{x} p^x (1-p)^{1-x} \forall x \in S$ Since $\sum_{i=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} = 1$ and We know that $0 \le \binom{n}{x} p^x (1-p)^{1-x} \forall x \in S$, it must be true that $0\binom{n}{x}p^x(1-p)^{1-x} \leq 1 \forall x \in S$ as well. Therefore $0 \leq \binom{n}{x}p^x(1-p)^{1-x} \leq 1 \forall x \in S$

$\square$

2. Mean
$$E[X] = np$$

Proof.

$$\begin{aligned}
E[X] &= \sum_{x \in S} x \binom{n}{x} p_X(x) \\
&= \sum_{x=0}^{n} x \left(\frac{n!}{x!(n-x)!}\right) p^x(1-p)^{n-x} \\
&= \sum_{x=1}^{n} x \left(\frac{n!}{x!(n-x)!}\right) p^x(1-p)^{n-x} \leftarrow \text{term where } x = 0 \text{ is ust } 0 \\
&= \sum_{x=1}^{n} \left(\frac{n!}{(x-1)!(n-x)!}\right) p^x(1-p)^{n-x} \leftarrow \text{Cancel out } x \text{ in numerator} \\
& \qquad \text{and denominator} \\
&= n\sum_{x=1}^{n} \left(\frac{(n-1)!}{(x-1)!(n-x)!}\right) p^x(1-p)^{n-x} \leftarrow \text{pull out a factor of } n \\
&= n\sum_{x=1}^{n} \left(\frac{(n-1)!}{(x-1)!((n-1)-(x-1))!}\right) p^x(1-p)^{n-x} \leftarrow \text{Rewriting } (n-x)! \text{ term in} \\
& \qquad \text{denominator as } ((n-1)-(x-1))! \\
&= np\sum_{x=1}^{n} \left(\frac{(n-1)!}{(x-1)!((n-1)-(x-1))!}\right) p^{x-1}(1-p)^{n-x} \leftarrow \text{pulling out a factor of } p \\
&= np\sum_{x=1}^{n} \left(\frac{(n-1)!}{(x-1)!((n-1)-(x-1))!}\right) p^{x-1}(1-p)^{(n-1)-(x-1)} \\
& \qquad \uparrow \text{Rewriting } n-x \text{ term in exponent as } (n-1)-(x-1) \\
&= np\sum_{x-1=0}^{n-1} \left(\frac{(n-1)!}{(x-1)!((n-1)-(x-1))!}\right) p^{x-1}(1-p)^{(n-1)-(x-1)} \\
& \qquad \uparrow \text{Rewriting } x = 1 \text{ lower boundary of summation as } x-1 = 0 \\
& \qquad \text{Note, this also changes the upper bound from n to n-1} \\
&= np\sum_{y=0}^{n-1} \left(\frac{(n-1)!}{(y!((n-1)-y)!}\right) p^{y}(1-p)^{(n-1)-(y)} \\
& \qquad \uparrow \text{recplacing } x-1 \text{ with } y \text{ in equation for readability} \\
&= np\sum_{y=0}^{n-1} \binom{n-1}{y} p^{y}(1-p)^{(n-1)-(y)} \leftarrow \text{Combination definition}
\end{aligned}$$

$$\begin{aligned}
&= np(1-p+p)^{n-1} \leftarrow \text{Binomial Expansion Theorem} \\
&= np1^{n-1} \\
&= np
\end{aligned}$$

□

3. Variance

$$V[X] = np(1-p)$$

Proof. proof left as an exercise □