

HW 11

Solutions

1. Let X be distributed $\text{Gamma}(\alpha, \beta)$ and let $a > 0$ be a constant. Find the distribution of $U = aX$.

Solution:

$$\begin{aligned}h(X) &= aX \\ \Rightarrow h^{-1}(U) &= U/a \\ \Rightarrow f_U(u) &= f_X(h^{-1}(u)) \left| \frac{d}{du} h^{-1}(u) \right|\end{aligned}$$

$$\begin{aligned}f_X(h^{-1}(u)) \left| \frac{d}{du} h^{-1}(u) \right| &= \frac{1}{\Gamma(\alpha)\beta^\alpha} (u/a)^{\alpha-1} e^{-\frac{(u/a)}{\beta}} \left| \frac{1}{a} \right| \\ &= \frac{1}{\Gamma(\alpha)\beta^\alpha} u^{\alpha-1} \left(\frac{1}{a}\right)^\alpha e^{-\frac{u}{a\beta}} \\ &= \frac{1}{\Gamma(\alpha)(a\beta)^\alpha} u^{\alpha-1} e^{-\frac{u}{a\beta}}\end{aligned}$$

Where $0 < u < \infty$.

Thus $U \sim \Gamma(\alpha, a\beta)$

2. Let X be distributed $\text{Normal}(\mu, \sigma^2)$ and let $a > 0, b > 0$ be constants. Find the distribution of $U = a(X + b)$.

Solution:

$$\begin{aligned}h(X) &= a(X + b) \\ \Rightarrow h^{-1}(U) &= \frac{U - ab}{a} \\ \Rightarrow f_U(u) &= f_X(h^{-1}(u)) \left| \frac{d}{du} h^{-1}(u) \right|\end{aligned}$$

$$f_X(h^{-1}(u)) \left| \frac{d}{du} h^{-1}(u) \right| = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{(U-ab)/a-\mu}{\sigma}\right)^2} \left| \frac{1}{a} \right|$$

$$\begin{aligned}
&= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{(U-ab)/a - \frac{a\mu}{a}}{\sigma}\right)^2} \frac{1}{a} \\
&= \frac{1}{\sqrt{2\pi}a\sigma} e^{-\frac{1}{2}\left(\frac{U-(b+\mu)}{a\sigma}\right)^2}
\end{aligned}$$

Where $-\infty < u < \infty$.

Thus $U \sim N(a(b+\mu), (a\sigma)^2)$

3. Let X be distributed $\text{Beta}(\alpha, \beta)$ and let $a > 0$ be a constant. Find the distribution of $U = aX$.

Solution:

$$\begin{aligned}
h(X) &= aX \\
\Rightarrow h^{-1}(U) &= U/a \\
\Rightarrow f_U(u) &= f_X(h^{-1}(u)) \left| \frac{d}{du} h^{-1}(u) \right|
\end{aligned}$$

$$\begin{aligned}
f_X(h^{-1}(u)) \left| \frac{d}{du} h^{-1}(u) \right| &= \frac{1}{B(\alpha, \beta)} (u/a)^{\alpha-1} (1 - (u/a))^{\beta-1} \left| \frac{1}{a} \right| \\
&= \frac{1}{B(\alpha, \beta)} \left(\frac{1}{a}\right)^{\alpha+\beta-1} u^{\alpha-1} (a-u)^{\beta-1}
\end{aligned}$$

Where $0 < u/a < 1 \Rightarrow 0 < u < a$.

4. Let X be distributed $\text{Uniform}(0, \theta)$ where $\theta > 0$ and let $a > 0$ be a constant. Find the distribution of $U = aX$.

Solution:

$$\begin{aligned}
F_U(u) &= P(U \leq u) \\
&= P(aX \leq u) \\
&= P\left(X \leq \frac{u}{a}\right) \\
&= \int_0^{u/a} \frac{1}{\theta} dx (= F_X(u/a)) \\
&= \frac{u}{a\theta}
\end{aligned}$$

Where $0 < u/a < \theta \Rightarrow 0 < u < a\theta$.

$$\begin{aligned}
\Rightarrow f_U(u) &= \frac{d}{du} F_U(u) \\
&= \frac{d}{du} \frac{u}{a\theta} \\
&= \frac{1}{a\theta}
\end{aligned}$$

Where $0 < u < a\theta$.

Thus $U \sim U(0, a\theta)$

5. Let X be distributed $\text{Normal}(\mu, \sigma^2)$. Find the distribution of $U = e^X$.

Solution:

$$\begin{aligned} h(X) &= e^X \\ \Rightarrow h^{-1}(U) &= \ln(U) \\ \Rightarrow f_U(u) &= f_X(h^{-1}(u)) \left| \frac{d}{du} h^{-1}(u) \right| \end{aligned}$$

$$\begin{aligned} f_X(h^{-1}(u)) \left| \frac{d}{du} h^{-1}(u) \right| &= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{\ln(u)-\mu}{\sigma}\right)^2} \left| \frac{1}{u} \right| \\ &= \frac{1}{u\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{\ln(u)-\mu}{\sigma}\right)^2} \end{aligned}$$

Where

$$\begin{aligned} -\infty < \ln(u) < \infty &\Rightarrow e^{-\infty} < u < e^{\infty} \\ &\Rightarrow 0 < u < \infty. \end{aligned}$$

This is called the lognormal distribution.

6. Let X_1, X_2 be distributed $\text{Bernoulli}(p)$, and let X_1 and X_2 be independent. Find the distribution of $U = X_1 + X_2$

Solution:

$$X_1, X_2 \sim \text{Bernoulli}(p) \Rightarrow m_{X_1}(t) = m_{X_2}(t) = 1 - p + pe^t$$

$$\begin{aligned} M_U(t) &= E[e^{Ut}] \\ &= E[e^{(X_1+X_2)t}] \\ &= E[e^{X_1t} e^{X_2t}] \\ &= E[e^{X_1t}] E[e^{X_2t}] \leftarrow \text{since } X_1 \text{ and } X_2 \text{ are independent} \\ &= (1 - p + pe^t)(1 - p + pe^t) \\ &= (1 - p + pe^t)^2 \end{aligned}$$

Thus $U \sim \text{Binomial}(2, p)$

7. Let X_1, X_2 be distributed $\text{Poisson}(\lambda)$, and let X_1 and X_2 be independent. Find the distribution of $U = X_1 + X_2$

Solution:

$$X_1, X_2 \sim \text{Poisson}(\lambda) \Rightarrow m_{X_1}(t) = m_{X_2}(t) = e^{\lambda(e^t-1)}$$

$$\begin{aligned} M_U(t) &= E[e^{Ut}] \\ &= E[e^{(X_1+X_2)t}] \\ &= E[e^{X_1t} e^{X_2t}] \\ &= E[e^{X_1t}] E[e^{X_2t}] \leftarrow \text{since } X_1 \text{ and } X_2 \text{ are independent} \\ &= e^{\lambda(e^t-1)} e^{\lambda(e^t-1)} \\ &= e^{(2\lambda)(e^t-1)} \end{aligned}$$

Thus $U \sim \text{Poisson}(2\lambda)$

8. Let X_1, X_2 be distributed $\text{Gamma}(\alpha, \beta)$, and let X_1 and X_2 be independent. Find the distribution of $U = X_1 + X_2$

Solution:

$$X_1, X_2 \sim \text{Gamma}(\alpha, \beta) \Rightarrow m_{X_1}(t) = m_{X_2}(t) = (1 - \beta t)^{-\alpha} \text{ for } t < \frac{1}{\beta}$$

$$\begin{aligned} M_U(t) &= E[e^{Ut}] \\ &= E[e^{(X_1+X_2)t}] \\ &= E[e^{X_1t} e^{X_2t}] \\ &= E[e^{X_1t}] E[e^{X_2t}] \leftarrow \text{since } X_1 \text{ and } X_2 \text{ are independent} \\ &= (1 - \beta t)^{-\alpha} (1 - \beta t)^{-\alpha} \\ &= (1 - \beta t)^{-(2\alpha)}, \text{ when } t < \frac{1}{\beta} \end{aligned}$$

Thus $U \sim \text{Gamma}(2\alpha, \beta)$

9. Let X be distributed $\text{Uniform}(0, 1)$. Find the distribution of $U = -\ln(X)$

Solution:

$$\begin{aligned} F_U(u) &= P(U \leq u) \\ &= P(-\ln(X) \leq u) \\ &= P(X > e^{-u}) \\ &= \int_{e^{-u}}^1 \frac{1}{1} dx \\ &= 1 - e^{-u} \end{aligned}$$

Where

$$\begin{aligned} 0 < x < 1 &\Rightarrow -\infty < \ln(x) < 0 \\ &\Rightarrow 0 < -\ln(x) < \infty \\ &\Rightarrow 0 < u < \infty \end{aligned}$$

$$\begin{aligned}
\Rightarrow f_U(u) &= \frac{d}{du} F_U(u) \\
&= \frac{d}{du} (1 - e^{-u}) \\
&= e^{-u}
\end{aligned}$$

Where $0 < u < \infty$.

Thus $U \sim \text{Exponential}(1)$

10. Let X_1, X_2, \dots, X_n all be distributed $\text{Normal}(\mu, \sigma^2)$. Let X_1, X_2, \dots, X_n be mutually independent. Find the distribution of $U = \sum_{i=1}^n X_i$

Solution:

$$X_1, X_2, \dots, X_n \sim \text{Normal}(\mu, \sigma) \Rightarrow m_{X_1}(t) = m_{X_2}(t) = \dots = m_{X_n}(t) = e^{\mu t + \sigma^2 \frac{t^2}{2}}$$

$$\begin{aligned}
M_U(t) &= E[e^{Ut}] \\
&= E[e^{(\sum_{i=1}^n X_i)t}] \\
&= E[e^{(\sum_{i=1}^n X_i t)}] \\
&= E[\prod_{i=1}^n e^{X_i t}] \\
&= \prod_{i=1}^n E[e^{X_i t}] \leftarrow \text{since } X_1, X_2, \dots, X_n \text{ are mutually independent} \\
&= \prod_{i=1}^n e^{\mu t + \sigma^2 \frac{t^2}{2}} \\
&= e^{(n\mu)t + (n\sigma^2) \frac{t^2}{2}}
\end{aligned}$$

Thus $U \sim \text{Normal}(n\mu, n\sigma^2)$

11. Problem 6.14 from the book (p 309)

Since Y_1 and Y_2 are independent, so $f(y_1, y_2) = 18(y_1 - y_1^2)y_2^2$, for $0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1$.

Let $U = Y_1 Y_2$. Then,

$$\begin{aligned}
F_U(u) &= P(U \leq u) = P(Y_1 Y_2 \leq u) = P(Y_1 \leq u / Y_2) = P(Y_1 > u / Y_2) = 1 - \int_u^1 \int_{u/y_2}^1 18(y_1 - y_1^2)y_2^2 dy_1 dy_2 \\
&= 9u^2 - 8u^3 + 6u^3 \ln u. \\
f_U(u) &= F'_U(u) = 18u(1 - u + u \ln u), \quad 0 \leq u \leq 1.
\end{aligned}$$

12. Problem 6.15 from the book (p 309)

Let U have a uniform distribution on $(0, 1)$. The distribution function for U is $F_U(u) = P(U \leq u) = u$, $0 \leq u \leq 1$. For a function G , we require $G(U) = Y$ where Y has distribution function $F_Y(y) = 1 - e^{-y^2}$, $y \geq 0$. Note that

$$F_Y(y) = P(Y \leq y) = P(G(U) \leq y) = P[U \leq G^{-1}(y)] = F_U[G^{-1}(y)] = u.$$

So it must be true that $G^{-1}(y) = 1 - e^{-y^2} = u$ so that $G(u) = [-\ln(1-u)]^{1/2}$. Therefore, the random variable $Y = [-\ln(U-1)]^{1/2}$ has distribution function $F_Y(y)$.

13. Problem 6.69 from the book (p 332)

a. The joint density is $f(y_1, y_2) = \frac{1}{y_1^2 y_2^2}$, $y_1 > 1, y_2 > 1$.

b. We have that $y_1 = u_1 u_2$ and $y_2 = u_2(1 - u_1)$. The Jacobian of transformation is u_2 . So,

$$f(u_1, u_2) = \frac{1}{u_1^2 u_2^3 (1-u_1)^2},$$

with limits as specified in the problem.

c. The limits may be simplified to: $1/u_1 < u_2$, $0 < u_1 < 1/2$, or $1/(1-u_1) < u_2$, $1/2 \leq u_1 \leq 1$.

d. If $0 < u_1 < 1/2$, then $f_{U_1}(u_1) = \int_{1/u_1}^{\infty} \frac{1}{u_1^2 u_2^3 (1-u_1)^2} du_2 = \frac{1}{2(1-u_1)^2}$.

If $1/2 \leq u_1 \leq 1$, then $f_{U_1}(u_1) = \int_{1/(1-u_1)}^{\infty} \frac{1}{u_1^2 u_2^3 (1-u_1)^2} du_2 = \frac{1}{2u_1^2}$.

e. Not independent since the joint density does not factor. Also note that the support is not rectangular.