Indepence & Expectation

1 Indepence

1.1 Motivation

- Much like how we discussed the concept of two events being independent of each other, we can also discuss the idea of random variables being independent
- When we say that two random variables are independent, we are conveying that the value of one of the random variables does not in any way affect the value of the other
- This is important for regression and prediction.
- If we can show that two measure are independent, then that means that one measurement will be useless a predictor of the value of the other measurement

1.2 Definition

- Let X_1, X_2 be Random variables with joint CDF F, and with marginal CDFs F_1 and F_2, respectively.

  Definition 1. *X_1, and X_2 are said to be* ***independent*** *if*

  $$F(x_1, x_2) = F_1(x_1)F_2(x_2)\forall x_1, x_2$$

  Definition 2. *If two random variables are not independent, then they are said to be* ***dependent***

- The concept of independence can be extended to more than two random variables in a similar manner under two different definitions : pairwise independent & mutualy indpendent

1.3 Theorems

- There are a few theorems that make showing the independence of random variables somewhat easier
- The first theorem has to do with when X_1 and X_2 are discrete

  Theorem 1. *Let X_1 and X_2 be discrete random variables with joint PDF p and marginal PDFs p_1 and p_2, respectively. Then X_1 and X_2 are independent iff*

  $$p(x_1, x_2) = p_1(x_1)p_2(x_2)\forall x_1, x_2$$

- The second theorem has to do with when X_1 and X_2 are continuous

Theorem 2. *Let X_1 and X_2 be continuous random variables with joint PDF f and marginal PDFs f_1 and f_2, respectively. Then X_1 and X_2 are independent iff*

$$f(x_1, x_2) = f_1(x_1)f_2(x_2)\forall x_1, x_2$$

- The third theorem gives us an extra tool when X_1 and X_2 are continuous
- The Theorem specifically applies when the support of X_1 and X_2 do not depend on each other, or in other words, when the joint PDF is positive over a (constant) square region of space

Theorem 3. *Let X_1 and X_2 be continuous random variables with joint PDF f. If for constant values a, b, c, d $(a < b, c < d)$ the pdf f is positive only when $a \leq x_1 \leq b$ and $c \leq x_2 \leq d$, and zero other wise, then X_1 and X_2 are independent iff*

$$f(x_1, x_2) = g(x_1)h(x_2)$$

For some functions g and h

- Basically, if the PDF is positive over a square rgion of space, then if the pdf can be written as the product of a function of x_1 and a (different) function of x_2, then X_1 and X_2 are independent

1.4 Examples

1. Let X_1 and X_2 be discrete random variables with the following joint pdf:

$$p(x_1, x_2) = \begin{cases} \frac{p^{x_2}(1-p)^{1-x_2}}{x_2+1} & \text{for} x_1 = 0, x_2 \text{ and } x_2 = 0, 1 \\ 0 & \text{else} \end{cases}$$

Where $0 < p < 1$. Let's find out if X_1 and X_2 are independent or dependent. We already know the marginal distribution of X_2 is

$$p_2(x_2) = \begin{cases} p^{x_2}(1-p)^{1-x_2} & x_2 = 0, 1 \\ 0 & \text{else} \end{cases}$$

This implies that $p_2(0) = 1 - p$ We can also see that

$$\begin{aligned} p_1(0) &= \sum_{x_2 \in S_2} p(0, x_2) \\ &= \sum_{x_2=0}^{1} p(0, x_2) \\ &= \sum_{x_2=0}^{1} p(0, x_2) \\ &= \sum_{x_2=0}^{1} \frac{p^{x_2}(1-p)^{1-x_2}}{x_2+1} \\ &= (1-p) + p/2 \end{aligned}$$

So, now we consider whether $p(0,0) = p_1(0)p_2(0)$

$$\begin{array}{rcl} p(0,0) & ? & p_1(0)p_2(0) \\ (1-p) & ? & [(1-p)+p/2][1-p] \\ (1-p) & \neq & (1-p)^2 + \dfrac{p(1-p)}{2} \end{array}$$

For any p between 0 and 1 (in fact the two side are equal only when $p = 1$, which isoutside the range of possible values). Thus X_1 and X_2 are dependent

2. Let X_1 and X_2 be continuous random variables with the following pdf:

$$f(x_1, x_2) = \begin{cases} exp(-x_1) & \text{for} 0 < x_1 \text{ and } 0 < x_2 < 1 \\ 0 & \text{else} \end{cases}$$

Let's find out if X_1 and X_2 are independent or dependent.

a) Solution 1:
We already know the marginal distribution of X_1 is

$$f_1(x_1) = \begin{cases} e^{-x_1} & 0 < x_1 \\ 0 & \text{else} \end{cases}$$

We can also find the marginal distribution of X_2 to be

$$\begin{array}{rcl} f_2(x_2) & = & \displaystyle\int_{-\infty}^{\infty} f(x_1, x_2)\mathrm{d}x_1 \\ & = & \begin{cases} \int_0^{\infty} exp(-x_1)\mathrm{d}x_1 & 0 < x_2 < 1 \\ 0 & \text{else} \end{cases} \\ & = & \begin{cases} -exp(-x_1)|_{x_1=0}^{\infty} & 0 < x_2 < 1 \\ 0 & \text{else} \end{cases} \\ & = & \begin{cases} \lim_{x_1 \to \infty} -exp(-x_1) - (-exp(-0)) & 0 < x_2 < 1 \\ 0 & \text{else} \end{cases} \\ & = & \begin{cases} 0 - (-1) & 0 < x_2 < 1 \\ 0 & \text{else} \end{cases} \\ & = & \begin{cases} 1 & 0 < x_2 < 1 \\ 0 & \text{else} \end{cases} \end{array}$$

We see that infact $f(x_1, x_2) = f_1(x_1)f_2(x_2)$ for all x_1, x_2, and thus X_1 and X_2 are independent

b) Solution 2:
We see that the joint PDF is positive only when $0 < x_1 < \infty$ and when $0 < x_2 < 1$, so we can appeal to our third theorem. If we let

$$g(x_1) = \begin{cases} exp(-x_1) & 0 < x_1 \\ 0 & \text{else} \end{cases}$$

and we let

$$h(x_2) = \begin{cases} 1 & 0 < x_2 < 1 \\ 0 & \text{else} \end{cases}$$

Then we see that $f(x_1, x_2) = g(x_1)h(x_2)$. Thus X_1 and X_2 are independent

2 Expectation of functions of multiple random variables

- Now, we will go over Expectation of functions of multiple random variables

2.1 Definitions

- As before, we will give seperate definitions based on groups of discrete random variables and groups of continuous random variables
- We will begin with discrete random variables

Definition 3. *Let $X_1, ..., X_n$ be discrete random variables with joint pdf p and let g be a real valued function defined over all possible values of $(X_1, ..., X_n)$. Then the expected value of $g(X_1, ..., X_n)$ is defined to be*

$$E[g(X_1, ..., X_n)] = \sum_{x_1 \in S_1} ... \sum_{x_n \in S_n} g(x_1, ..., x_n)p(x_1, ..., x_n)$$

- An now when the random variables are continuous

Definition 4. *Let $X_1, ..., X_n$ be continuous random variables with joint pdf f and let g be a real valued function defined over all possible values of $(X_1, ..., X_n)$. Then the expected value of $g(X_1, ..., X_n)$ is defined to be*

$$E[g(X_1, ..., X_n)] = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} g(x_1, ..., x_n)f(x_1, ..., x_n)\mathrm{d}x_1 ... \mathrm{d}x_n$$

2.2 Theorems

- As one might expect, our expectation theorems can be extended into multivariate expectation

Theorem 4. *Let $X_1, ..., X_n$ be random variables and let c be a real valued constant. Then*

$$E[c] = c$$

Theorem 5. *Let $X_1, ..., X_n$ be random variables, let c be a real valued constant and let g be a real valued function defined over all possible values of $(X_1, ..., X_n)$. Then*

$$E[cg(X_1, ..., X_n)] = cE[g(X_1, ..., X_n)]$$

Theorem 6. *Let $X_1, ..., X_n$ be random variables and let $g_1 ... g_k$ be real valued functions defined over all possible values of $(X_1, ..., X_n)$. Then*

$$E[\sum_{i=1}^{k} g_i(X_1, ..., X_n)] = \sum_{i=1}^{k} E[g_i(X_1, ..., X_n)]$$

- We also have a Theorem for when the random variables are independent

Theorem 7. *Let X_1, X_2 be random variables and let g and h be real values functionsdefined over the supports of X_1, and X_2, respectively. If X_1 and X_2 are independent, then*

$$E[g(X_1)h(X_2)] = E[g(X_1)]E[h(X_2)]$$

Here we will do the proof when X_1 and X_2 are continuous with pdf f and marginal PDFs f_1 and f_2. The proof when the random variablesare discrete is similar

Proof.

$$\begin{aligned}
E[g(X_1)h(X_2)] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1)h(X_2)f(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2 \leftarrow \text{By Definition} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1)h(x_2)f_1(x_1)f_2(x_2)\mathrm{d}x_1\mathrm{d}x_2 \leftarrow \text{Because } X_1 \text{ and } X_2 \text{ are independent} \\
&= \int_{-\infty}^{\infty} h(x_2)f_2(x_2)[\int_{-\infty}^{\infty} g(x_1)f_1(x_1)\mathrm{d}x_1]\mathrm{d}x_2 \\
&= \int_{-\infty}^{\infty} h(x_2)f_2(x_2)E[g(X_1)]\mathrm{d}x_2 \leftarrow \text{By Definition} \\
&= E[g(X_1)]\int_{-\infty}^{\infty} h(x_2)f_2(x_2)\mathrm{d}x_2 \\
&= E[g(X_1)]E[h(X_2)] \leftarrow \text{By Definition}
\end{aligned}$$

□

2.3 Examples

1. Let X_1 and X_2 be discrete random variables with the following joint pdf:

$$p(x_1,x_2) = \begin{cases} \frac{p^{x_2}(1-p)^{1-x_2}}{x_2+1} & \text{for} x_1 = 0, x_2 \text{ and } x_2 = 0,1 \\ 0 & \text{else} \end{cases}$$

Where $0 < p < 1$. Let's find $E[X_1X_2]$.

$$E[X_1X_2] = \sum_{x_1 \in S_1}\sum_{x_2 \in S_2} x_1x_2p(x_1,x_2)$$

$$\begin{aligned}
&= \sum_{x_2=0}^{1}\sum_{x_1=0}^{x_2} x_1 x_2 p(x_1,x_2) \\
&= (0)(0)p(0,0) + (0)(1)p(0,1) + (1)(1)p(1,1) \\
&= p(1,1) \\
&= \frac{p^1(1-p)^{1-1}}{1+1} \\
&= p/2
\end{aligned}$$

2. Let X_1 and X_2 be continuous random variables with the following pdf:

$$f(x_1,x_2) = \begin{cases} exp(-x_1) & \text{for} 0 < x_1 \text{ and } 0 < x_2 < 1 \\ 0 & \text{else} \end{cases}$$

Let's find $E[X_1^2X_2 - 2X_1X_2 + X_2]$

$$\begin{aligned}
E[X_1^2X_2 - 2X_1X_2 - X_2] &= E[(X_1^2 - 2X_1 + 1)(X_2)] \\
&= E[(X_1^2 - 2X_1 + 1)]E[(X_2)] \leftarrow \text{Since we have shown that } X_1 \text{ and } X_2 \text{ ar} \\
&= E[(X_1-1)^2]E[(X_2)] \\
&= V[X_1]E[(X_2)] \leftarrow \text{Ww know from before that } X_1 \sim Exp(1) \\
&= (1^2)(\frac{1}{2}) \\
&= 1/2
\end{aligned}$$

Since We know that $X_1 \sim Exp(1) \Rightarrow V[X_1] = 1^2$ and we also know that $X_2 \sim U(0,1) \Rightarrow E[X_2] = \frac{1+0}{2}$