1 Motivation

- We began this section of the course discussing the situation in which we take multiple measurements at the same time and would like to model how the relate to each other
- In these cases sometimes we would like to look at the underlying distribution of just one of those measurments, Other times we wantto look at the distribution of a measurement, given the observation of a different measurement
- The first concept is encapsulated in what we call the *Marginal Distribution* of a random variable, while the second concept is addressed with what we call the *Conditional Distribution*

2 Definitions

- While, it is possible to find marginal and conditional distributions of more than one random variable at a time (i.e. we have a joint distribution of many random variable and we want the marginal or conditional distribution of some of them), we will focus on defining marginal and conditional distributions when we are working with two random variables at a time
- So, with that being said, let X_1, X_2 be random variables with supports S_1, S_2
- For this section, we will assume that X_1, X_2 are both the same kind of random variable
- In other words, we will assume that X_1, X_2 are either both discrete, or are both continuous

2.1 Marginal Distributions

- First we will give the definition when X_1, X_2 are both discrete

  Definition 1. *Let p be the joint pdf of X_1, X_2. Then the marginal distribution of X_1 is*
  $$p_1(x_1) = \sum_{x_2 \in S_2} p(x_1, x_2)$$

- Now we will give the definition when X_1, x_2 are continuous

  Definition 2. *Let f be the joint pdf of X_1, X_2. the the marginal distribution of X_1 is*
  $$f_1(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2) \mathrm{d}x_2$$

- Note, when you are looking for a marginal distribution, the random variable that you select to be X_1 and which one you select to be X_2 will depend on the question you are trying to answer

2.2 Conditional Distributions

- Now, we will give the definition of the conditinal distribution when X_1, X_2 are both discrete

  Definition 3. *Let p be the joint pdf of X_1, X_2, and let p_2 be the marginal pdf of X_2. Then the conditional distribution of X_1 given $X_2 = x_2$ (denoted $X_1|X_2 = x_2$), where x_2 is in the support of X_2, is*

  $$p_{1|2}(x_1) = \frac{p(x_1, x_2)}{p_2(x_2)}$$

- And when X_1, X_2 are continuous the definition is as follows:

  Definition 4. *Let f be the joint pdf of X_1, X_2, and let f_2 be the marginal pdf of X_2. Then the conditional distribution of X_1 given $X_2 = x_2$ (denoted $X_1|X_2 = x_2$), where x_2 is in the support of X_2, is*

  $$f_{1|2}(x_1) = \frac{f(x_1, x_2)}{f_2(x_2)}$$

3 Notes

- Note, both marginal and conditional distributions are still distributions in their own right.
- This means that they adhere to all of the properties that PDFs do
- This also means that we can talk about the expectations of these distributions

4 Examples

1. Let X_1 and X_2 be discrete random variables with the following joint pdf:

   $$p(x_1, x_2) = \begin{cases} \frac{p^{x_2}(1-p)^{1-x_2}}{x_2+1} & \text{for} x_1 = 0, x_2 \text{ and } x_2 = 0, 1 \\ 0 & \text{else} \end{cases}$$

   Where $0 < p < 1$.

a) Let's find the marginal distribution of X_2

$$\begin{aligned}
p_2(x_2) &= \sum_{x_1 \in S_1} p(x_1, x_2) \\
&= \sum_{x_1=0}^{x_2} p(x_1, x_2) \\
&= \begin{cases} \sum_{x_1=0}^{0} 1-p & x_2 = 0 \\ \sum_{x_1=0}^{1} \frac{p}{2} & x_2 = 1 \\ 0 & \text{else} \end{cases} \\
&= \begin{cases} 1-p & x_2 = 0 \\ \frac{p}{2} + \frac{p}{2} & x_2 = 1 \\ 0 & \text{else} \end{cases} \\
&= \begin{cases} p^{x_2}(1-p)^{1-x_2} & x_2 = 0, 1 \\ 0 & \text{else} \end{cases}
\end{aligned}$$

So, the marginal distribution of X_2 is $X_2 \sim Bern(p)$

b) Now lets find the distribution of $X_1|X_2 = x_2$

$$\begin{aligned}
p_{1|2}(x_1) &= \frac{p(x_1, x_2)}{p_2(x_2)} \\
&= \begin{cases} \left(\frac{p^{x_2}(1-p)^{1-x_2}}{x_2+1}\right) / p^{x_2}(1-p)^{x_2} & x_1 = 0, x_2 \\ 0 & \text{else} \end{cases} \\
&= \begin{cases} \frac{1}{x_2+1} & x_1 = 0, x_2 \\ 0 & \text{else} \end{cases}
\end{aligned}$$

2. Let X_1 and X_2 be continuous random variables with the following pdf:

$$f(x_1, x_2) = \begin{cases} exp(-x_1) & \text{for} 0 < x_1 \text{ and } 0 < x_2 < 1 \\ 0 & \text{else} \end{cases}$$

a) Let's find the marginal distribution of X_1

$$\begin{aligned}
f_1(x_1) &= \int_{-\infty}^{\infty} f(x_1, x_2) \mathrm{d}x_2 \\
&= \begin{cases} \int_0^1 e^{-x_1} \mathrm{d}x_2 & 0 < x_1 \\ 0 & \text{else} \end{cases} \\
&= \begin{cases} e^{-x_1} x_2 |_{x_2=0}^{1} & 0 < x_1 \\ 0 & \text{else} \end{cases} \\
&= \begin{cases} e^{-x_1} & 0 < x_1 \\ 0 & \text{else} \end{cases}
\end{aligned}$$

So, the marginal distribution of X_1 is $Exp(1)$

b) Now, lets find the conditional distribution of $X_2|X_1 = x_1$

$$\begin{aligned} f_{2|1}(x_2) &= \frac{f(x_1, x_2)}{f_1(x_1)} \\ &= \begin{cases} \frac{exp(-x_1)}{exp(-x_1)} & 0 < x_2 < 1 \\ 0 & \text{else} \end{cases} \\ &= \begin{cases} 1 & 0 < x_2 < 1 \\ 0 & \text{else} \end{cases} \end{aligned}$$

So, the conditional distiribution of $X_2|X_1 = x_1$ is $U(0, 1)$

3. Let $(X_1, X_2) \sim N_2(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$. Here we will simply state the marginal and conditional distributions of X_1, X_2, $X_1|X_2 = x_2$, and $X_2|X_1 = x_1$, and leave the derivations for another time
   a) The marginal distiribution of X_1 is $N(\mu_1, \sigma_1^2)$
   b) The marginal distiribution of X_2 is $N(\mu_2, \sigma_2^2)$
   c) The conditional Distribution of $X_1|X_2 = x_2$ is $N(\mu_1 + \frac{\rho\sigma_1}{\sigma_2}(x_2 - \mu_2), (1 - \rho^2)\sigma_1^2)$
   d) The conditional Distribution of $X_2|X_1 = x_1$ is $N(\mu_2 + \frac{\rho\sigma_2}{\sigma_1}(x_1 - \mu_1), (1 - \rho^2)\sigma_2^2)$