Moment Generating functions (MGFs)

1 Definitions

- Before we can discuss what a moment generating function is, we must first define what we mean by a *moment*

  Definition 1. *Let k be a non-negative integer, and let X be a random variable with support S and PDF $p_X(x)$. Then the* $\boldsymbol{k^{th}}$ ***moment*** *of X is $E[X^k]$*

- We have already worked with the first moment, $E[X]$

  Definition 2. *The* ***Moment Generating Function*** *(or just* ***MGF*** *for short) of a Random Variable X with support S and PDF p_X is defined to be*

  $$M_X(t) = E[e^{tX}]$$

  where there exists a $b > 0$ such that $M_X(t) < \infty$ for $|t| < b$

- So, why is this called the moment generating functuion?
- Lets consider the taylor exapnsion of e^{Xt}

$$e^{Xt} = \sum_{i=0}^{\infty} \frac{(Xt)^i}{i!}$$

- So taking the expectation on both sides of this equation we see

$$\begin{aligned} E[e^{Xt}] &= E[\sum_{i=0}^{\infty} \frac{(Xt)^i}{i!}] \\ &= \sum_{i=0}^{\infty} E[\frac{(Xt)^i}{i!}] \\ &= \sum_{i=0}^{\infty} \frac{t^i}{i!} E[X^i] \end{aligned}$$

- So, the moment generating function is really a sum of terms involving all of the moments of X
- This means that if we differentiate the MGF with respect to t and then set t to zero, we will be left with just a moment of X

- For example:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E[e^{Xt}] &= \frac{\mathrm{d}}{\mathrm{d}t}E[\sum_{i=0}^{\infty}\frac{(Xt)^i}{i!}] \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=0}^{\infty}\frac{t^i}{i!}E[X^i]\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(1+\sum_{i=1}^{\infty}\frac{t^i}{i!}E[X^i]\right) \leftarrow \text{Pulled out the first term of the sum} \\
&= \left(0+\sum_{i=1}^{\infty}\frac{it^{i-1}}{i!}E[X^i]\right) \\
&= \left(0+\frac{1t^{1-1}}{i!}E[X^1]+\sum_{i=2}^{\infty}\frac{it^{i-1}}{i!}E[X^i]\right) \leftarrow \text{Pulled out the first term of the sum} \\
&= \left(0+E[X]+\sum_{i=2}^{\infty}\frac{it^{i-1}}{i!}E[X^i]\right) \\
&= \left(0+E[X]+\sum_{i=2}^{\infty}\frac{i(0)^{i-1}}{i!}E[X^i]\right) \leftarrow \text{Set } t=0 \\
&= \left(0+E[X]+\sum_{i=2}^{\infty}0\right) \leftarrow \text{Set } t=0 \\
&= E[X]
\end{aligned}
$$

- If we were to take further derivatives, we would be able to get higher order moments
- We can sumarize this characteristic as follows

Theorem 1. *Let X be a Random Variable with support S, PDF $p_X(x)$ and MGF $M_X(t)$. Then*

$$E[X^k] = \left[\frac{\mathrm{d}^k}{\mathrm{d}t^k}M_X(t)\right]_{t=0}$$

- While we will not prove it in this class, it turns out that the MGF of a Random variable is unique
- That is to say that if a Random variable has a particular PDF, then that implies a particular MGF and vice a versa.

Theorem 2. *Let X be a Random Variable with support S. Then, assuming that both the PDF and MGF exist for the distribution of X, X has one unique PDF p_X and one unique MGF $M_X(t)$*

- This means that we can identify the distribution of a random variable based on the formula for the MGF in the same way we can identify the distribution of the Random Variable based on its PDF

2 Examples

- Now we will derive some moment generating functions

1. Let $X \sim Bern(p)$. Derive $M_X(t)$, the MGF of X
Solution:

$$\begin{aligned}
M_X(t) &= E[e^{Xt}] \\
&= \sum_{x \in S} e^{xt} p_X(x) \\
&= \sum_{x=0}^{1} e^{xt} p^x (1-p)^{1-x} \\
&= (1-p)e^{(0)t} + pe^t \\
&= (1-p) + pe^t
\end{aligned}$$

2. Let $X \sim Nbinomial(r, p)$. Derive $M_X(t)$, the MGF of X
Solution:

$$\begin{aligned}
M_X(t) &= E[e^{Xt}] \\
&= \sum_{x \in S} e^{xt} p_X(x) \\
&= \sum_{x=r}^{\infty} \binom{x-1}{r-1} e^{xt} p^r (1-p)^{x-r} \\
&= \sum_{x=r}^{\infty} \binom{x-1}{r-1} e^{(r+x-r)t} p^r (1-p)^{x-r} \\
&= \sum_{x=r}^{\infty} \binom{x-1}{r-1} e^{rt} e^{(x-r)t} p^r (1-p)^{x-r} \\
&= \sum_{x=r}^{\infty} \binom{x-1}{r-1} (pe^t)^r [(1-p)e^t]^{x-r} \\
&= (pe^t)^r \frac{[1-(1-p)e^t]^r}{[1-(1-p)e^t]^r} \sum_{x=r}^{\infty} \binom{x-1}{r-1} [(1-p)e^t]^{x-r}
\end{aligned}$$

$$= \frac{(pe^t)^r}{[1-(1-p)e^t]^r} \sum_{x=r}^{\infty} \binom{x-1}{r-1} [1-(1-p)e^t]^r [(1-p)e^t]^{x-r}$$

Note, in the definition of the MGF, $M_X(t)$ only needs to exist for $|t| < b$ for some positive b. If we restrict t to be $|t| < -\ln(1-p)$, then we see that $t < -\ln(1-p)$, which implies that $e^t < e^{-\ln(1-p)} = \frac{1}{1-p}$. Thus $0 < (1-p)e^t < 1$ and $0 < [1-(1-p)e^t] < 1$. This means that $\sum_{x=r}^{\infty} \binom{x-1}{r-1} [1-(1-p)e^t]^r [(1-p)e^t]^{x-r} = 1$, giving us our MGF of

$$M_X(t) = \left(\frac{(pe^t)}{1-(1-p)e^t} \right)^r \text{ when } |t| < -\ln(1-p)$$

3. Let $X \sim poisson(\lambda)$. Derive $M_X(t)$, the MGF of X (Derivation left as HW)

$$M_X(t) = exp(\lambda(e^t - 1))$$

Note: $exp(x) = e^x$. i.e. $exp()$ is just another way of writing $e^()$

3 Exercise

1. Let $X \sim bin(n,p)$. Derive $M_X(t)$, the MGF of X

4 Solution

1. Let $X \sim bin(n,p)$. Show that $M_X(t)$, the MGF of X, is $(pe^t + 1 - p)^n$
   Solution:

$$\begin{aligned}
M_X(t) &= E[e^{Xt}] \\
&= \sum_{x \in S} e^{xt} p_X(x) \\
&= \sum_{x=0}^{n} \binom{n}{x} e^{xt} p^x (1-p)^{n-x} \\
&= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x} \\
&= (pe^t + 1 - p)^n \leftarrow \text{From binomial Expansion theorem}
\end{aligned}$$