Continuous Random Variables

1 Definition

- Now that we have discussed Discrete Random Variables we will now move on to continuous Random Variables
- Like Discrete Random Variables, continuous Random Variables are numerical representations of the outcomes of a probability experiment
- Where Discrete Random Variables have a support with finite or countable infinite number of elelements, the support of a continuous random variable has an uncountably infinite number of elements
- Typically this means that the support of a continuous random variable is composed of intervals of numbers
- Because there are an uncountably infinite number of possible values of a continuous random variable, we see that the probability of the random variable being any one particular value must be zero
- i.e. $P(X = x) = 0 \forall x \in S$, the support of the continuous random variable X
- This is true because if an uncountably infinite number of potential values of the random variable had some non zero probability, then the total probability would be greater than 1
- another way to think of this is imagine you are waiting for a bus that is scheduled to arrive at noon
  - At first it seems reasonable to say that there is some probability that the bus will be on time
  - But when we say ”on time” do we mean exactly noon?
  - Usually if the bus stops at 1 second after noon we would still call this on time
  - When we think about it, it is really unlikely that the bus will arrive exactly at noon (not even a millisecond off!)
- This means that our original Probability Distribution function definition doesn't make sense for continuous random variables
- Instead we will look at the probability that the random variable is in a range of values, specifically we will look at $P(X \leq x)$

  Definition 1. *The* Cumulative Probability Distribution Function *(or just cumulative distribution function; CDF for short) of a (any, discrete or continuous) random variable is the function F_X that take possible values of the random variable and returns the probability that the random variable is less tha nor equal to that number. i.e. $F_X = P(X \leq x) \forall x$.*

- Properties of the CDF of a random variable X
  - $F_X(-\infty) = \lim_{x\to-\infty} F_X(x) = 0$
  - $F_X(\infty) = \lim_{x\to\infty} F_X(x) = 1$
  - F_X is a non-decresing function (i.e. for $x_1 < x_2$, $F_X(x_1) < F_X(x_2)$)
- Note, both discrete and continuous random variables have CDFs
- CDFs of Discrete random variables will have jumps in values
  - For example consider X, a discrete Random Variable with $P(X = 1) = .2$ and $P(X = 2) = .8$
  - Then the CDF of X will be

$$F_X(x) = \begin{cases} 0 & \text{if } x < 1 \\ .2 & \text{if } 1 \le x < 2 \\ 1 & \text{if } 2 \le x \end{cases}$$

- CDFs of continuous Random variables on the other hand will be smooth, continuous functions (i.e. no jumps)
- This difference is another way you can think of differentiating discrete and Continous Random Variables
- We would still like a function like the Probability distribution function of the discrete random variable to help us characterize the distribution of a con tinuous random varibale
- So, for continous random variables we introduce the following definition of probability distribution functions:

  Definition 2. *The Proability Distribution function of a Continuous Random variable X with CDF F_X (sometimes also called a probability densifty function, PDF for short) is defined to be the function f_X such that*

$$f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_X(x) = F'_X(x)$$

- Conversely we can say that if a Random Variable has a PDF $f_X(x)$, then the following theorem gives us the CDF of the random Variable:

  Theorem 1. *Let X be a continuous Random Variable with PDF f_X. The the CDF of the distribution, $F_X(x)$ is*

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\mathrm{d}y$$

- Like with the properties of a valid discrete random variable's PDF, there are two properties of a valid continuous random Variable PDF

- Let X be a continuous Random Variable with support S and PDF f_X. Then,
  1. $f_X(x) \geq 0 \forall x \in S$
  2. $\int_{-\infty}^{\infty} f_X(x) \mathrm{d}x = 1$

2 Future Questions and definitions

- When we get to the statistical part of the course (next semester), we will want to be able to answer questions about Rando variables that are of certain forms.
- Here we will introduce the forms of these questions and define the related answers
  1. Suppose We have a Random Variable X (discrete or continuous). Sometimes we will want to find the value ☺ such that $P(X \leq ☺) = p$, where p is a value between 0 and 1.
     - This question is asking for a *quantile* of the distribution of X. Formally, we define a quantile as follows:
       Definition 3. *Let X be a Random variable with CDF $F_X(x)$ and let p be a value between 0 and 1. The p^{th} quantile of the distribution of X, ϕ_p is the smallest value such that $P(X \leq \phi_p) = F_X(\phi_p) \geq p$.*
     - Sometimes the p^{th} quantile is also refered to as the $(100 \times p)^{th}$ percentile
     - Note that, because continuous Random Variables have continuous CDFs, the p^{th} quantile (ϕ_p) of a continuous random variable will be the smallest value such that $P(X \leq) = F_X(\phi_p) = p$
  2. Again suppose that we have a Random Variable X. Sometimes we will want to know the probability that the random variable will be between two real values a and b, where $a < b$ [$P(a \leq X \leq b)$].
     - For Discrete Random variables, We simply take every value that X can take on that is between a and b and take the sum of the probabilities of those numbers
     - For continuous Random Variables with PDF f_X we simply use the following theorem:
       Theorem 2. *Let X be a Continuous Random Variable with pdf f_X. Then,*

$$P(a \leq X \leq b) = \int_a^b f_X(x) \mathrm{d}x$$

Proof. Before we prove the theorem, we will need to prove a lemma first.

Lemma 1. *Let X be a continuous Random variable and let a be a real valued constant. Then*

$$P(X \leq a) = P(X < a)$$

Proof.

$$
\begin{aligned}
P(X \le a) &= P(X < a) + P(X = a) \leftarrow \text{Mutually Exclusive Events} \\
&= P(X < a) \leftarrow P(X = a) = 0 \text{ for any constant } a
\end{aligned}
$$

□

$$
\begin{aligned}
P(a \le X \le b) &= P((a \le X) \cap (X \le b)) \\
&= P(a \le X) + P(X \le b) - P((a \le X) \cup (X \le b)) \\
&\quad \uparrow \text{General additive rule} \\
&= 1 - P(X < a) + P(X \le b) - P((a \le X) \cup (X \le b)) \\
&\quad \uparrow \text{Compliment rule} \\
&= 1 - P(X \le a) + P(X \le b) - P((a \le X) \cup (X \le b)) \\
&\quad \uparrow \text{From Lemma} \\
&= 1 - P(X \le a) + P(X \le b) - P(-\infty < X < \infty) \\
&\quad \uparrow X \le b \cup a \le X \text{ is equivalent to } X \text{ being any number since } a < b \\
&= 1 - P(X \le a) + P(X \le b) - 1 \\
&= P(X \le b) - P(X \le a) \\
&= \int_{-\infty}^{b} f_X(x)\mathrm{d}x - \int_{-\infty}^{a} f_X(x)\mathrm{d}x \\
&= \int_{a}^{b} f_X(x)\mathrm{d}x
\end{aligned}
$$

□

3 Expectation of a Continuous Random Variable

- Fos continuous Random Variables we also have expectation, but we define it differently

  Definition 4. *Let X be a continuous Random Variable with PDF $f_X(x)$ and let g be a real valued function thati defined over the set of real numbers. Then the expectation of $g(X)$ is*

  $$E[g(X)] = \int_{\infty}^{\infty} g(x) f_X(x)\mathrm{d}x$$

- Like with discrete Random variables, when g is just the identity function (i.e. $g(y) = y$), then $E[g(X)] = E[X]$ is the *Expected Value* of the distribution of X

- And like with discrete Random Variables, when g is the function $g(☺) = (☺ - E[X])^2$, then $E[g(X)] = E[(X - E[X])^2]$ is the *variance* of the distribution of X

4 Expectation Theorems for Continuous Random Variables

- Let X be a continuous RV with PDF f_X
- Let c be a real valued constant. Then,

$$E[c] = c$$

Proof.

$$\begin{aligned} E[c] &= \int_{-\infty}^{\infty} c f_X(x)\mathrm{d}x \\ &= c\int_{-\infty}^{\infty} f_X(x)\mathrm{d}x \leftarrow \text{Pulling out a factor of } c \\ &= c \leftarrow \text{By Second rule of Continuous PDFs} \end{aligned}$$

□

- Let c be a real valued constant and let g be a real valued function that is defined over the support of X. Then,

$$E[cg(X)] = cE[g(X)]$$

Proof.

$$\begin{aligned} E[cg(X)] &= \int_{-\infty}^{\infty} cg(X) f_X(x)\mathrm{d}x \\ &= c\int_{-\infty}^{\infty} g(X) f_X(x)\mathrm{d}x \leftarrow \text{Pulling out a factor of } c \\ &= cE[g(X)] \leftarrow \text{Definition of } E[g(X)] \end{aligned}$$

□

- Let $g_1, g_2, g_3, ..., g_n$ be n real valued functions that are all defined over the support of X. Then,

$$E[\sum_{i=1}^{n} g_i(X)] = \sum_{i=1}^{n} E[g_i(X)]$$

Proof.

$$\begin{aligned}
E[\sum i = 1^n g_i(X)] &= \int_{-\infty}^{\infty} \sum_{i=1}^{n} g_i(X) f_X(x) \mathrm{d}x \\
&= \sum_{i=1}^{n} \int_{-\infty}^{\infty} g_i(X) f_X(x) \mathrm{d}x \leftarrow \text{Integral of the sums is the sum ofth integrals} \\
&= \sum_{i=1}^{n} E[g_i(X)] \leftarrow \text{Defintion of } E[g_i(X)]
\end{aligned}$$

□

- Variance Equality

$$V[X] = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

Note1: $(E[X])^2$ is often denoted as $E^2[X]$ *Note2:* This proof is identical to the proof for the varaince of discrete random variables

Proof.

$$\begin{aligned}
V[X] &= E[(X - E[X])^2] \\
&= E[X^2 - 2E[X] + E^2[X]] \\
&= E[X^2]E[-2XE[X]] + E[E^2[X]] \leftarrow \text{By the Third Expectation Rule} \\
&= E[X^2] - 2E[X]E[X] + E^2[X] \leftarrow \text{By the First and second Expectation Rules since} \\
&\qquad -2E[X] \text{ and } E^2[X] \text{ are both constant values} \\
&= E[X^2] - 2E^2[X] + E^2[X] \\
&= E[X^2] - E^2[X]
\end{aligned}$$

□

5 Exercises

Let X be a continuous Random Variable with PDF $f_X(x)$ and let a and b be real, constant values. Prove that

1. $E[aX + b] = aE[X] + b$
2. $V[aX + b] = a^2 V[X]$

6 Solutions

1. $E[aX+b] = aE[X] + b$
   Solution:

$$\begin{aligned}
E[aX+b] &= \int_{-\infty}^{\infty} (ax+b) f_X(x) \mathrm{d}x \\
&= \int_{-\infty}^{\infty} ax f_X(x) \mathrm{d}x + \int_{\infty}^{\infty} b f_X(x) \mathrm{d}x \\
&= a \int_{-\infty}^{\infty} x f_X(x) \mathrm{d}x + b \int_{\infty}^{\infty} f_X(x) \mathrm{d}x \\
&= aE[X] + b
\end{aligned}$$

2. $V[aX+b] = a^2 V[X]$

$$\begin{aligned}
V[aX+b] &= E[(aX+b)^2] - E^2[aX+b] \\
\text{and} & \\
E[(aX+b)^2] &= \int_{-\infty}^{\infty} (ax+b)^2 f_X(x) \mathrm{d}x \\
&= \int_{-\infty}^{\infty} (a^2x^2 + 2abx + b^2) f_X(x) \mathrm{d}x \\
&= \int_{-\infty}^{\infty} a^2x^2 f_X(x) \mathrm{d}x + \int_{\infty}^{\infty} 2abx f_X(x) \mathrm{d}x + \int_{-\infty}^{\infty} b^2 f_X(x) \mathrm{d}x \\
&= a^2 \int_{-\infty}^{\infty} x^2 f_X(x) \mathrm{d}x + 2ab \int_{\infty}^{\infty} x f_X(x) \mathrm{d}x + b^2 \int_{-\infty}^{\infty} f_X(x) \mathrm{d}x \\
&= a^2 E[X^2] + 2abE[X] + b^2 \\
\text{So,} & \\
V[aX+b] &= E[(aX+b)^2] - E^2[aX+b] \\
&= a^2 E[X^2] + 2abE[X] + b^2 - (a^2 E^[X] + 2abE[X] + b^2) \\
&= a^2 (E[X^2] - E^2[X]) \\
&= a^2 V[X]
\end{aligned}$$

7 Moment Generating Function of a Continuous Random Variable

- Like With discrete random variables, the definiton of a moment for the distirbution of a continuous random variable depends on the expectation of the distribution of the Random variable:

Definition 5. *Let k be a non-negative integer, and let X be a continuous random variable with PDF $f_X(x)$. Then the* ***k^{th} moment*** *of X is $E[X^k] = \int_{\infty}^{\infty} x^k f_X(x) \mathrm{d}x$*

- Not surprisingly, the definition of of the moeement Generating function for the distirbution of a continuous random variable is also similar to that of the discrete random variable distribution:

Definition 6. *The* ***Moment Generating Function*** *(or just* ***MGF*** *for short) of a Continuous Random Variable X with PDF f_X is defined to be*

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x) \mathrm{d}x$$

where there exists a $b > 0$ such that $M_X(t) < \infty$ for $|t| < b$

- and this Momentgenerating function has the exact same properties that the Moment Generating Function of Discerte Random variables do, namely

Theorem 3. *Let X be a continous Random Variable with PDF $f_X(x)$ and MGF $M_X(t)$. Then*

$$E[X^k] = \left[\frac{\mathrm{d}^k}{\mathrm{d}t^k} M_X(t)\right]_{t=0}$$