The Beta Distirbution

1 Definition & Motivation

- Our last continuous Distribution is the Beta distribution
- The Beta distribution is very useful for modeling bounded random variables with a non uniform distribution of probability
- The Beta distribution is particularly useful in studying Bayestian statistics

Definition 1. *We say that the continuous Random Variable X has a Beta distribution with parameters α and β when X has the PDF*

$$f_X(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & 0 < x < 1 \\ 0 & \textit{else} \end{cases}$$

Where $0 < \alpha$, $0 < \beta$. We denote this $X \sim Beta(\alpha, \beta)$ (Note the support of X is $S = (0, 1)$).

2 The Beta Function

- To Help us with the Verification, mean and variance derivations we will make use of what we call the Beta function:

Definition 2. *The Beta function $B(a, b)$ is a function of two positive values a and b, and is defined to be:*

$$B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1} \mathrm{d}x$$

- We we also use the following theorem (whose proof we leave for te end of the lecture):

Theorem 1. *Let a and b be positive real values. Then,*

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

- Because of this theorem, often times the pdf of $X \sim Beta(\alpha, \beta)$ is simplified to be:

$$f_X(x) = \begin{cases} \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} & 0 < x < 1 \\ 0 & \text{else} \end{cases}$$

3 Verify, Mean, Variance

Let $X \sim Beta(\alpha, \beta)$

1. Verify

   a) $f_X(x) \geq 0 \forall x \in S$

   Proof. We see that $\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} > 0$ by definition and since $0 < x < 1$ it is true that $0 < 1 - x < 1$. This in turn implies that $0 < x^{\alpha-1} < 1$ and $0 < (1-x)^{\beta-1} < 1$. Since $f_X(x)$ is the product of these three positve terms when $x > 0$ and $f_X(x) = 0$ other wise, we see that

   $$f_X(x) \geq 0$$

   □

   b) $\int_{-\infty}^{\infty} f_X(x) \mathrm{d}x = 1$

   Proof.

   $$\begin{aligned}
   \int_{-\infty}^{\infty} f_X(x)\mathrm{d}x &= \int_{-\infty}^{0} f_X(x)\mathrm{d}x + \int_{0}^{1} f_X(x)\mathrm{d}x + \int_{1}^{\infty} f_X(x)\mathrm{d}x \\
   &= 0 + \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\mathrm{d}x + 0 \\
   &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\mathrm{d}x \\
   &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} B(\alpha,\beta) \leftarrow \text{Definition of Beta function} \\
   &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \leftarrow \text{Beta function Theorem} \\
   &= 1
   \end{aligned}$$

   □

2. Mean

   $$E[X] = \frac{\alpha}{\alpha+\beta}$$

Proof.

$$\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} x f_X(x)\mathrm{d}x \\
&= \int_{-\infty}^{0} x f_X(x)\mathrm{d}x + \int_{0}^{1} x f_X(x)\mathrm{d}x + \int_{1}^{\infty} x f_X(x)\mathrm{d}x \\
&= 0 + \int_0^1 x\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}\mathrm{d}x + 0 \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 x^{\alpha}(1-x)^{\beta-1}\mathrm{d}x \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 x^{(\alpha+1)-1}(1-x)^{\beta-1}\mathrm{d}x \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}B(\alpha+1,\beta) \leftarrow \text{Definition of Beta function} \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} \leftarrow \text{Beta function Theorem} \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\alpha\Gamma(\alpha)\Gamma(\beta)}{(\alpha+\beta)\Gamma(\alpha+\beta)} \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\frac{\alpha}{\alpha+\beta} \\
&= \frac{\alpha}{\alpha+\beta}
\end{aligned}$$

□

3. Variance

$$V[X] = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$$

Proof. Proof left as HW □

4 The Beta Function Theorem Proof

Here we will present the proof of the Beta function Theorem, which states that for all positive real values a and b,

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Proof. Let a and b be real positive values and consider the following:

$$\Gamma(a)\Gamma(b) = \left(\int_0^{\infty} u^{a-1}e^{-u}\mathrm{d}u\right)\left(\int_0^{\infty} v^{b-1}e^{-v}\mathrm{d}v\right) \leftarrow \text{Definition of the Gamma function}$$

$$
\begin{aligned}
&= \int_0^\infty \int_0^\infty u^{a-1}e^{-u}v^{b-1}e^{-v}\mathrm{d}v\mathrm{d}u \\
&= \int_0^\infty \int_0^\infty u^{a-1}v^{b-1}e^{-(u+v)}\mathrm{d}v\mathrm{d}u
\end{aligned}
$$

Next we will make use of the following lemma:

Lemma 1. *Let f be an integrable function defined over D, region of R^2. Suppose we wish to evaluate the integral*

$$\iint_D f(x,y)\mathrm{d}x\mathrm{d}y$$

If for each x and y pair in D, we let z and t be real values such that $x = g(z,t)$ and $y = h(z,t)$ for integrable functions g and h then it can be shown that

$$\iint_D f(x,y)\mathrm{d}x\mathrm{d}y = \iint_S f(g(z,t),h(z,t))|J|\mathrm{d}z\mathrm{d}t$$

Where J is the determinant of the jacobian matrix

$$
\begin{aligned}
J &= det\begin{pmatrix} \frac{\partial g(z,t)}{\partial z} & \frac{\partial h(z,t)}{\partial z} \\ \frac{\partial g(z,t)}{\partial t} & \frac{\partial h(z,t)}{\partial t} \end{pmatrix} \\
&= (\frac{\partial g(z,t)}{\partial z})(\frac{\partial h(z,t)}{\partial t}) - (\frac{\partial h(z,t)}{\partial z})(\frac{\partial g(z,t)}{\partial t})
\end{aligned}
$$

and S is the transformation of the region D (in other words, S is the set of all points such that if you put those points in the functions g and h you get the set of points that comprise the region D).

So, to use our lemma we will let $u = g(t,z) = zt$ and we will let $v = h(t,z) = z(1-t)$. This means that our Jacobian will be

$$\begin{pmatrix} t & 1-t \\ z & -z \end{pmatrix}$$

And thus

$$
\begin{aligned}
|J| &= |(t)(-z) - (1-t)(z)| \\
&= |-z| \\
&= z
\end{aligned}
$$

Additionally, Since in our case D is the region bounded by $0 < u$ and $0 < v$, S will be the region bounded by $0 < z$ and $0 < t < 1$, which we conclude from the following logic:

$$
\begin{aligned}
0 < zt &\Rightarrow 0 < z \text{ and } 0 < t \\
0 < z(1-t) &\Rightarrow 0 < 1-t \\
&\Rightarrow t < 1
\end{aligned}
$$

Therefore

$$\begin{aligned}
\int_0^\infty \int_0^\infty u^{a-1} v^{b-1} e^{-(u+v)} \mathrm{d}v \mathrm{d}u &= \iint\limits_S f(g(z,t), h(z,t)) |J| \mathrm{d}z \mathrm{d}t \\
&= \int_0^\infty \int_0^1 (zt)^{a-1} (z(1-t))^{b-1} e^{-((zt)+(z(1-t)))} z \mathrm{d}t \mathrm{d}z \\
&= \int_0^\infty \int_0^1 z^{a+b-2+1} t^{a-1} (1-t)^{b-1} e^{-z} \mathrm{d}t \mathrm{d}z \\
&= \int_0^\infty z^{a+b-1} e^{-z} \mathrm{d}z \int_0^1 t^{a-1} (1-t)^{b-1} \mathrm{d}t \\
&= \Gamma(a+b) \int_0^1 t^{a-1} (1-t)^{b-1} \mathrm{d}t
\end{aligned}$$

So, this means that we have shown that

$$\begin{aligned}
\Gamma(a)\Gamma(b) &= \Gamma(a+b) \int_0^1 t^{a-1} (1-t)^{b-1} \mathrm{d}t \\
\Rightarrow \int_0^1 t^{a-1} (1-t)^{b-1} \mathrm{d}t &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \\
\Rightarrow B(a,b) &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}
\end{aligned}$$

Thus concluding our proof.

$\square$