Homework 5

Solutions!

1. A trial has just resulted in a hung jury because eight members of the jury were in favor of a guilty verdict and the other four were for acquittal. If the jurors leave the jury room in random order and each of the first four jurors leaving the room is accosted by a reporter in hopes of getting an interview, what is the distribution, mean, and variance of X = the number of jurors favoring acquittal among those accosted? When naming the distribution of X, be sure to include the parameter values for that particular distribution.

   Solution:
   $X \sim Hypergeo(N = 12, r = 4, n = 4)$ Where

   - N is the population size (here, the number of jurors)
   - r is the number of population members with the desired attribute (here, the number of jurors who voted for acquittal)
   - n is the sample size (here, our sample is the first four people to leave the room and be accosted by reporters)

   and

$$\begin{aligned} E[X] &= \frac{nr}{N} \\ &= \frac{(4)(4)}{(12)} \\ &= \frac{4}{3} \end{aligned}$$

$$\begin{aligned} V[X] &= n\left(\frac{r}{N}\right)\left(\frac{N-r}{N}\right)\left(\frac{N-n}{N-1}\right) \\ &= (4)\left(\frac{(4)}{(12)}\right)\left(\frac{(12)-(4)}{(12)}\right)\left(\frac{(12)-(4)}{(12)-1}\right) \\ &= (4)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\left(\frac{8}{11}\right) \\ &= \frac{64}{99} \end{aligned}$$

2. Suppose that you and your family go on a safari in Africa. As you are driving around the savanna, the guide informs you that visitors usually see one lion per day, on average. Let X be the number of lions you observe on your trip today. Please identify the distribution (including parameters), mean and variance of X.

   Solution:
   $X \sim Pois(\lambda = 1)$ Where λ is the average number of lions seen on a given day by visitors, and

$$\begin{aligned} E[X] &= \lambda \\ &= 1, \end{aligned}$$

$$\begin{aligned} V[X] &= \lambda \\ &= 1. \end{aligned}$$

3. Let $X \sim Hypergeo(N, r, n)$ where $N, r,$ and n are positive integers such that $n \leq r < N$ and $n \leq N - r$.

   Derive the variance of X (i.e. show that $V[X] = n(\frac{r}{N})(\frac{N-r}{N})(\frac{N-n}{N-1})$)
   Solution:

$$\begin{aligned}
V[X] &= E[X^2] - E^2[X] \\
\text{also,} & \\
E[(X-1)X] &= E[X^2] - E[X] \\
\implies E[X^2] &= E[(X-1)X] + E[X] \\
&= E[(X-1)X] + \frac{nr}{N} \\
E[(X-1)X] &= \sum_{x \in S} (x-1)x p_X(x) \\
&= \sum_{x=0}^{n} (x-1)x \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}} \\
&= \sum_{x=2}^{n} (x-1)x \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}} \leftarrow \text{First two terms are just 0} \\
&= \sum_{x=2}^{n} (x-1)x \frac{r!}{x!(r-x)!} \frac{\binom{N-r}{n-x}}{\binom{N}{n}} \\
&= r(r-1) \sum_{x=2}^{n} \frac{(r-2)!}{(x-2)!(r-x)!} \frac{\binom{N-r}{n-x}}{\binom{N}{n}} \\
&= r(r-1) \sum_{x-2=0}^{n-2} \frac{\binom{r-2}{x-2}\binom{N-r}{(n-2)-(x-2)}}{\binom{N}{n}}
\end{aligned}$$

$$
\begin{aligned}
&= r(r-1)\sum_{y=0}^{n-2}\frac{\binom{r-2}{y}\binom{N-r}{(n-2)-y}}{\binom{N}{n}} \leftarrow \text{Relabel } x-2 \text{ as } y \\
&= r(r-1)\frac{\binom{(N-r)+(r-2)}{(n-2)}}{\binom{N}{n}} \leftarrow \text{From Theorem 1} \\
&= r(r-1)\frac{\binom{N-2}{n-2}}{\binom{N}{n}} \\
&= r(r-1)\left(\frac{(N-2)!}{(n-2)!(N-n)!}\right)/\left(\frac{N!}{n!(N-n)!}\right) \\
&= r(r-1)\frac{(N-2)!n!(N-n)!}{N!(n-2)!(N-n)!} \\
&= \frac{n(n-1)r(r-1)}{N(N-1)} \\
\implies E[X^2] &= E[(X-1)X] + \frac{nr}{N} \\
&= \frac{n(n-1)r(r-1)}{N(N-1)} + \frac{nr}{N} \\
&= \frac{n(n-1)r(r-1)}{N(N-1)} + \frac{nr(N-1)}{N(N-1)} \\
&= \frac{n(n-1)r(r-1)+nr(N-1)}{N(N-1)} \\
&= \frac{nr[(n-1)(r-1)+(N-1)]}{N(N-1)} \\
\implies V[X] &= \frac{nr[(n-1)(r-1)+(N-1)]}{N(N-1)} - \frac{n^2r^2}{N^2} \\
&= \frac{nr[(n-1)(r-1)+(N-1)]N}{N^2(N-1)} - \frac{n^2r^2(N-1)}{N^2(N-1)} \\
&= \frac{nr[(n-1)(r-1)+(N-1)]N - n^2r^2(N-1)}{N^2(N-1)} \\
&= \frac{nr[(n-1)(r-1)N+(N-1)N - nr(N-1)]}{N^2(N-1)} \\
&= n\frac{r}{N}\frac{[Nnr - nN - rN + N + N^2 - N - nrN + nr]}{N(N-1)} \\
&= n\frac{r}{N}\frac{[N^2 - nN - rN + nr]}{N(N-1)} \\
&= n\frac{r}{N}\frac{(N-r)(N-n)}{N(N-1)} \\
&= n\left(\frac{r}{N}\right)\left(\frac{N-r}{N}\right)\left(\frac{N-n}{N-1}\right)
\end{aligned}
$$

4. Let $X \sim Poisson(\lambda)$ where $\lambda > 0$.

   Derive the variance of X (i.e. show that $V[X] = \lambda$)
   Solution:

$$
\begin{aligned}
V[X] &= E[X^2] - E^2[X] \\
\text{also,} & \\
E[(X-1)X] &= E[X^2] - E[X] \\
\Rightarrow E[X^2] &= E[(X-1)X] + E[X] \\
&= E[(X-1)X] + \lambda \\
E[(X-1)X] &= \sum_{x \in S} (x-1)x p_X(x) \\
&= \sum_{x=0}^{\infty} (x-1)x \frac{\lambda^x e^{-\lambda}}{x!} \\
&= \sum_{x=2}^{\infty} (x-1)x \frac{\lambda^x e^{-\lambda}}{x!} \leftarrow \text{First two terms are just 0} \\
&= \sum_{x=2}^{\infty} \frac{\lambda^x e^{-\lambda}}{(x-2)!} \\
&= \lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2} e^{-\lambda}}{(x-2)!} \\
&= \lambda^2 \sum_{x-2=0}^{\infty} \frac{\lambda^{x-2} e^{-\lambda}}{(x-2)!} \\
&= \lambda^2 \sum_{y=0}^{\infty} \frac{\lambda^y e^{-\lambda}}{(y)!} \leftarrow \text{Relabeling } x-2=y \\
&= \lambda^2(1) \leftarrow \text{We have already shown that } \sum_{y=0}^{\infty} \frac{\lambda^y e^{-\lambda}}{y!} = 1 \\
&= \lambda^2 \\
\Rightarrow E[X^2] &= E[(X-1)X] + E[X] \\
&= \lambda^2 + \lambda \\
\Rightarrow V[X] &= E[X^2] - E^2[X] \\
&= \lambda^2 + \lambda - \lambda^2 \\
&= \lambda
\end{aligned}
$$

5. Let $X \sim Geo(p)$ where $0 < p < 1$.
   Without referring to the MGF of the Negative Binomial, Derive the MGF of X (i.e. show that $M_X(t) = \frac{(pe^t)}{1-(1-p)e^t}$ when $|t| < -\ln(1-p)$)

Solution:

$$\begin{aligned}
M_X(t) &= E[e^{Xt}] \\
&= \sum_{x\in S} e^{xt} p_X(x) \\
&= \sum_{x=1}^{\infty} e^{xt} p(1-p)^{x-1} \\
&= \sum_{x=1}^{\infty} e^{(1+x-1)t} p(1-p)^{x-1} \\
&= \sum_{x=1}^{\infty} e^{t} e^{(x-1)t} p(1-p)^{x-1} \\
&= \sum_{x=1}^{\infty} pe^{t}[(1-p)e^t]^{x-1} \\
&= pe^t \frac{[1-(1-p)e^t]}{[1-(1-p)e^t]} \sum_{x=1}^{\infty}[(1-p)e^t]^{x-1} \\
&= \frac{(pe^t)}{[1-(1-p)e^t]} \sum_{x=1}^{\infty}[1-(1-p)e^t][(1-p)e^t]^{x-1}
\end{aligned}$$

Note, in the definition of the MGF, $M_X(t)$ only needs to exist for $|t| < b$ for some positive b. If we restrict t to be $|t| < -\ln(1-p)$, then we see that $t < -\ln(1-p)$, which implies that $e^t < e^{-\ln(1-p)} = \frac{1}{1-p}$. Thus $0 < (1-p)e^t < 1$ and $0 < [1-(1-p)e^t] < 1$. This means that $\sum_{x=1}^{\infty}[1-(1-p)e^t][(1-p)e^t]^{x-1} = 1$, giving us our MGF of

$$M_X(t) = \frac{(pe^t)}{1-(1-p)e^t} \text{ when } |t| < -\ln(1-p)$$

6. Let $X \sim Poisson(\lambda)$ where $\lambda > 0$.
   Derive the MGF of X (i.e. show that $M_X(t) = exp(\lambda(e^t - 1))$)
   Note: $exp(x) = e^x$. i.e. $exp()$ is just another way of writing $e^{()}$
   Solution:

$$
\begin{aligned}
M_X(t) &= E[e^{Xt}] \\
&= \sum_{x \in S} e^{xt} p_X(x) \\
&= \sum_{x=0}^{\infty} e^{xt} \frac{\lambda^x e^{-\lambda}}{x!} \\
&= \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x e^{-\lambda}}{x!} \\
&= \frac{exp(-\lambda e^t)}{exp(-\lambda e^t)} e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} \\
&= \frac{e^{-\lambda}}{exp(-\lambda e^t)} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x exp(-\lambda e^t)}{x!} \leftarrow \frac{(\lambda e^t)^x exp(-\lambda e^t)}{x!} \text{ pdf of } Poiss(\lambda e^t)... \\
&= \frac{e^{-\lambda}}{exp(-\lambda e^t)} \leftarrow \text{..sums to 1!} \\
&= exp(-\lambda + \lambda e^t) \\
&= exp(\lambda(e^t - 1))
\end{aligned}
$$