Homework 4

Solutions!

1. For each of the following situations, identify the random variable described and the corresponding distribution (including the parameter values), mean, and variance of the random variable described.

   a) Let's say that 1 in 10 metal objects buried in the sand is actually a piece of gold. Suppose that you like to spend your Saturdays at the beach with a metal detector looking for gold. As you comb the beach, whenever your metal detector goes off, you dig up whatever what the detector found. Once you find a piece of gold, you take note of how many things you dug up and go home to celebrate.

   Solution:

   $X = \#$ of times that you dig something up before going home.

$$\begin{aligned} X &\sim Geo(p = 0.1) \\ E[X] &= 1/(0.1) \\ &= 10 \\ V[X] &= (0.9)/(0.1)^2 \\ &= (0.9)(100) \\ &= 90 \end{aligned}$$

   b) The taste test for PTC (phenylthiocarbamide) is a favorite exercise in beginning human genetics classes. It has been established that a single gene determines whether or not an individual is a "taster". It is generally accepted that 70% of Americans are "tasters". Pretend are in a class of 20 students, and the class conducts this exercise and counts how many students in the class are "tasters".

   Solution:

   $X = \#$ of student in the class who are "tasters".

$$\begin{aligned} X &\sim Bin(n = 20, p = 0.7) \\ E[X] &= (20)(.7) \end{aligned}$$

$$
\begin{aligned}
&= 14 \\
V[X] &= (20)(.7)(.3) \\
&= 4.2
\end{aligned}
$$

2. Let $X \sim Bin(n, p)$ where n is a positive integer larger than 1 (we have already shown the case when $n = 1$) and $0 < p < 1$. Derive the varaince of X (i.e. show that $V[X] = np(1-p)$)
Solution:

$$
\begin{aligned}
V[X] &= E[X^2] - E^2[X] \\
\text{also,} & \\
E[(X(X-1)] &= E[X^2] - E[X] \\
\Rightarrow E[X^2] &= E[(X(X-1)] + E[X] \\
E[X(X-1)] &= \sum_{x \in S} x(x-1) p_X(x) \\
&= \sum_{x=0}^{n} x(x-1) \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=2}^{n} x(x-1) \binom{n}{x} p^x (1-p)^{n-x} \\
& \uparrow \text{ because terms where } x = 0 \text{ and } x = 1 \text{ are just } 0 \\
&= \sum_{x=2}^{n} x(x-1) \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \\
&= \sum_{x=2}^{n} \frac{n!}{(x-2)!(n-x)!} p^x (1-p)^{n-x} \\
&= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!(n-x)!} p^{x-2} (1-p)^{n-x} \\
&= n(n-1)p^2 \sum_{x-2=0}^{n-2} \frac{(n-2)!}{(x-2)!(n-x)!} p^{x-2} (1-p)^{n-x} \\
& \uparrow \text{ relabeling bounds of summation} \\
&= n(n-1)p^2 \sum_{x-2=0}^{n-2} \frac{(n-2)!}{(x-2)!((n-2)-(x-2))!} p^{x-2} (1-p)^{(n-2)-(x-2)} \\
&= n(n-1)p^2 \sum_{y=0}^{m} \frac{(m)!}{(y)!((m)-(y))!} p^{y} (1-p)^{(m)-(y)} \\
& \uparrow \text{ relabeling } x - 2 = y \text{ and } n - 2 = m \\
&= n(n-1)p^2 \\
& \uparrow \text{ We have already shown that } \textstyle\sum_{y=0}^{m} \frac{(m)!}{(y)!((m)-(y))!} p^{y} (1-p)^{(m)-(y)} = 1
\end{aligned}
$$

$$\begin{aligned}
\Rightarrow E[X^2] &= E[(X(X-1)] + E[X] \\
&= n(n-1)p^2 + np \\
\Rightarrow V[X] &= E[X^2] - E^2[X] \\
&= n(n-1)p^2 + np - n^2p^2 \\
&= n^2p^2 - np^2 + np - n^2p^2 \\
&= np(1-p)
\end{aligned}$$

3. Problem 3.55 from the book (p 113)

   Solution:

$$E\{Y(Y-1)Y-2)\} = \sum_{y=0}^{n} \frac{y(y-1)(y-2)n!}{y!(n-y)!} p^y (1-p)^{n-y} = \sum_{y=3}^{n} \frac{n(n-1)(n-2)(n-3)!}{(y-3)!(n-3-(y-3))!} p^y (1-p)^{n-y}$$

$$= n(n-1)(n-2)p^3 \sum_{z=0}^{n-3} \binom{n-3}{z} p^z (1-p)^{n-3-z} = n(n-1)(n-2)p^3 \ .$$

   Equating this to $E(Y^3) - 3E(Y^2) + 2E(Y)$, it is found that
   $E(Y^3) = 3n(n-1)p^2 - n(n-1)(n-2)p^3 + np.$

4. Problem 3.71 from the book (p 119)

   Solution:

   a. $P(Y > a) = \sum_{y=a+1}^{\infty} q^{y-1} p = q^a \sum_{x=1}^{\infty} q^{x-1} p = q^a$.

   b. From part a, $P(Y > a+b \mid Y > a) = \dfrac{P(Y > a+b, Y > a)}{P(Y > a)} = \dfrac{P(Y > a+b)}{P(Y > a)} = \dfrac{q^{a+b}}{q^a} = q^b$.

   c. The results in the past are not relevant to a future outcome (independent trials).

5. Let $X \sim NBin(r,p)$ where r is a positive integer and $0 < p < 1$. Derive the mean of X (i.e. show that $E[X] = \frac{r}{p}$)

   Solution:

$$\begin{aligned}
E[X] &= \sum_{x \in S} x p_X(x) \\
&= \sum_{x=r}^{\infty} x \binom{x-1}{r-1} p^r (1-p)^{x-r} \\
&= \sum_{x=r}^{\infty} x \frac{(x-1)!}{(r-1)!((x-1)-(r-1))!} p^r (1-p)^{x-r} \\
&= \frac{rp}{rp} \sum_{x=r}^{\infty} \frac{x!}{(r-1)!(x-r)!} p^r (1-p)^{x-r}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{r}{p}\sum_{x=r}^{\infty}\frac{x!}{r!(x-r)!}p^{r+1}(1-p)^{x-r} \\
&= \frac{r}{p}\sum_{x+1=r+1}^{\infty}\frac{x!}{r!(x-r)!}p^{r+1}(1-p)^{x-r} \\
&\quad \uparrow \text{shifting bounds of summation} \\
&= \frac{r}{p}\sum_{y=m}^{\infty}\frac{((y-1)!}{(m-1)!((y-1)-(m-1))!}p^{m}(1-p)^{(y-1)-(m-1)} \\
&\quad \uparrow \text{relabeling } x+1=y \text{ and } r+1=m \\
&= \frac{r}{p}\sum_{y=m}^{\infty}\binom{y-1}{m-1}p^{m}(1-p)^{y-m} \\
&= \frac{r}{p} \\
&\quad \uparrow \text{We have already shown that } \textstyle\sum_{y=m}^{\infty}\binom{y-1}{m-1}p^{m}(1-p)^{y-m}=1
\end{aligned}
$$

6. Let $X \sim NBin(r,p)$ where r is a positive integer and $0 < p < 1$. Derive the variance of X (i.e. show that $V[X] = \frac{r(1-p)}{p^2}$)
Solution:

$$
\begin{aligned}
V[X] &= E[X^2] - E^2[X] \\
\text{also,} & \\
E[(X(X+1)] &= E[X^2] + E[X] \\
\Rightarrow E[X^2] &= E[(X(X+1)] - E[X] \\
E[X(X+1)] &= \sum_{x\in S} x(x+1)p_X(x) \\
&= \sum_{x=r}^{\infty} x(x+1)\binom{x-1}{r-1}p^r(1-p)^{x-r} \\
&= \sum_{x=r}^{\infty} x(x+1)\frac{(x-1)!}{(r-1)!((x-1)-(r-1))!}p^r(1-p)^{x-r} \\
&= \frac{r(r+1)p^2}{r(r+1)p^2}\sum_{x=r}^{\infty}\frac{(x+1)!}{(r-1)!(x-r)!}p^r(1-p)^{x-r} \\
&= \frac{r(r+1)}{p^2}\sum_{x=r}^{\infty}\frac{(x+1)!}{(r+1)!(x-r)!}p^{r+2}(1-p)^{x-r} \\
&= \frac{r(r+1)}{p^2}\sum_{x+2=r+2}^{\infty}\frac{(x+1)!}{(r+1)!(x-r)!}p^{r+2}(1-p)^{x-r} \\
&\quad \uparrow \text{shifting bounds of summation} \\
&= \frac{r(r+1)}{p^2}\sum_{y=m}^{\infty}\frac{((y-1)!}{(m-1)!((y-2)-(m-2))!}p^{m}(1-p)^{(y-2)-(m-2)}
\end{aligned}
$$

$$
\begin{aligned}
& \uparrow \text{relabeling } x+2=y \text{ and } r+2=m \\
&= \frac{r(r+1)}{p^2} \sum_{y=m}^{\infty} \binom{y-1}{m-1} p^m (1-p)^{y-m} \\
&= \frac{r(r+1)}{p^2} \\
& \uparrow \text{We have already shown that } \textstyle\sum_{y=m}^{\infty} \binom{y-1}{m-1} p^m (1-p)^{y-m} = 1 \\
\Rightarrow E[X^2] &= E[(X(X+1)] - E[X] \\
&= \frac{r(r+1)}{p^2} - \frac{r}{p} \\
\Rightarrow V[X] &= E[X^2] - E^2[X] \\
&= \frac{r(r+1)}{p^2} - \frac{r}{p} - \frac{r^2}{p^2} \\
&= \frac{r^2+r}{p^2} - \frac{rp}{p^2} - \frac{r^2}{p^2} \\
&= \frac{r(1-p)}{p^2}
\end{aligned}
$$