The Gamma Function

1 Definition

- Here we will go over the Gamma **Function**, a function used to verify the Gamma distribution and the Normal distribution.

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  Definition 1. *The* Gamma Function *is the function* Γ *(defined for non-negative values) such that*
  $$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}\mathrm{d}x$$

- This will be useful in solving integrals of the form $\int_0^\infty x^{t-1}e^{-x}\mathrm{d}x$

- The Gamma function also has a useful property summarized in the following theorem:

  Theorem 1. *For all* $t > 1$
  $$\Gamma(t) = (t-1)\Gamma(t-1)$$

  Proof. Let $t > 1$. The by using integration by parts we see that
  $$\begin{aligned}
  \Gamma(t) &= \int_0^\infty x^{t-1}e^{-x}\mathrm{d}x \\
  u = x^{t-1} & \quad\quad v = -e^{-x} \\
  \mathrm{d}u = (t-1)x^{t-2}\mathrm{d}x & \quad\quad \mathrm{d}v = e^{-x}\mathrm{d}x \\
  \Gamma(t) &= [-x^{t-1}e^{-x}]_{x=0}^{\infty} - \int_0^\infty -e^{-x}(t-1)x^{t-2}\mathrm{d}x \\
  &= [-x^{t-1}e^{-x}]_{x=0}^{\infty} + (t-1)\int_0^\infty e^{-x}x^{t-2}\mathrm{d}x \\
  &= [-x^{t-1}e^{-x}]_{x=0}^{\infty} + (t-1)\Gamma(t-1) \\
  &= \left(\lim_{x\to\infty}[-x^{t-1}e^{-x}] - 0\right) + (t-1)\Gamma(t-1) \\
  &= \left(\lim_{x\to\infty} -x^{t-1}e^{-x}\right) + (t-1)\Gamma(t-1)
  \end{aligned}$$

  So, We must evaluate $\lim_{x\to\infty} -x^{t-1}e^{-x}$ (and show that it is equal to 0). We know that
  $$\lim_{x\to\infty} -x^{t-1}e^{-x} = -\lim_{x\to\infty} x^{t-1}e^{-x}$$

If $\lim_{x\to\infty} x^{t-1}e^{-x}$ exists. So we consider

$$\begin{aligned}\lim_{x\to\infty} x^{t-1}e^{-x} &= \frac{\lim_{x\to\infty} x^{t-1}}{\lim_{x\to\infty} e^{-x}}\\ &= \frac{\infty}{\infty}\end{aligned}$$

So, $\lim_{x\to\infty} \frac{x^{t-1}}{e^x}$ is of indeterminant form. This means we can apply L'Hospital's Rule to evaluate $\lim_{x\to\infty} \frac{x^{t-1}}{e^x}$. So,

$$\begin{aligned}\lim_{x\to\infty} \frac{x^{t-1}}{e^x} &= \lim_{x\to\infty} \frac{(t-1)x^{t-2}}{e^x}\\ &= \frac{\infty}{\infty}\end{aligned}$$

This is still of an indeterminant form. If we apply L'Hospital's Rule we will still have the same problem because we have x raised to a positive power going to ∞, but if we apply L'Hospital's Rule enough times then we will have x raised to a negative power, which will then go to 0. Specifically we want to apply L'Hospital's Rule enough times so that the power on x is negative. This means we want to apply the rule the smallest integer value larger than $t-1$ number of time. We will call this value $T = \lceil t-1 \rceil$, where $\lceil x \rceil$ is what you get when you round x up (note $T \geq t-1$). So,

$$\begin{aligned}\lim_{x\to\infty} \frac{x^{t-1}}{e^x} &= \lim_{x\to\infty} \frac{x^{t-1}}{e^x}\\ &\quad \text{L'Hospital's Rule once}\\ &= \lim_{x\to\infty} \frac{(t-1)x^{t-2}}{e^x}\\ &\vdots \quad \text{L'Hospital's Rule } T-1 \text{ more times } (T \text{ total times})\\ &= \lim_{x\to\infty} \frac{(t-1)...((t-1)-(T-1))x^{(t-1)-T}}{e^x}\\ &= \lim_{x\to\infty} \frac{(t-1)...((t-1)-(T-1))}{x^{T-(t-1)}e^x} \leftarrow \text{Since } T \geq t-1\\ &= 0\end{aligned}$$

So, We see that this implies that

$$\Gamma(t) = (t-1)\Gamma(t-1)$$

and thus our proof is shown. $\square$

- Note that this property has a particular implication when t is an integer:

  Theorem 2. *Let t be a positive integer. Then:*

  $$\Gamma(t) = (t-1)!$$

  Proof.

  $$\begin{array}{rcl}\Gamma(t) & = & (t-1)\Gamma(t-1) \leftarrow \text{Apply Theorem 1} \\ & = & (t-1)(t-2)\Gamma(t-2) \leftarrow \text{Apply Theorem 1 again} \\ & \vdots & \text{Apply Theorem 1 } t-3 \text{ more times (total of } t-1 \text{ applications)} \\ & = & (t-1)(t-2)...(t-(t-1))\Gamma(t-(t-1)) \\ & = & (t-1)(t-2)...(1)\Gamma(1) \\ & = & (t-1)(t-2)...(1) \\ & = & (t-1)!\end{array}$$

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- One other property about the Gamma Function that we will need (and simply state without proof) is summarized in the following theorem:

  Theorem 3. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

2 Exercises

Evaluate the following Integrals

1. $\int_0^\infty x^5 e^{-x} \mathrm{d}x$
2. $\int_0^\infty x^{11} e^{-x^2} \mathrm{d}x$
3. $\int_0^\infty e^{-x^2} \mathrm{d}x$

3 Solutions

1. $\int_0^\infty x^5 e^{-x} \mathrm{d}x$
   Solution:

$$\begin{array}{rcl}\displaystyle\int_0^\infty x^5 e^{-x} \mathrm{d}x & = & \displaystyle\int_0^\infty x^{6-1} e^{-x} \mathrm{d}x \\ & = & \Gamma(6) \\ & = & 5!\end{array}$$

2. $\int_0^\infty x^{11} e^{-x^2} \mathrm{d}x$
Solution:
Integrating by using Substituion we see

$$\begin{aligned}
\int_0^\infty x^{11} e^{-x^2} \mathrm{d}x &= \int_0^\infty x^{10} e^{-x^2} x \mathrm{d}x \\
u = x^2 & \quad \mathrm{d}u = 2x\mathrm{d}x \\
&\Rightarrow \quad x\mathrm{d}x = \frac{\mathrm{d}u}{2} \\
\Rightarrow u \text{ Ranges from } 0 \text{ to } \infty & \quad \text{When } x \text{ ranges from } 0 \text{ to } \infty \\
\Rightarrow \int_0^\infty x^{11} e^{-x^2} \mathrm{d}x &= \int_0^\infty u^5 e^{-u} \frac{\mathrm{d}u}{2} \\
&= \frac{1}{2} \int_0^\infty u^{6-1} e^{-u} \mathrm{d}u \\
&= \frac{1}{2} \Gamma(6) \\
&= \frac{1}{2} 5!
\end{aligned}$$

3. $\int_0^\infty e^{-x^2} \mathrm{d}x$
Solution:
Integrating by using Substituion we see

$$\begin{aligned}
u = x^2 & \quad \mathrm{d}u = 2x\mathrm{d}x \\
&\Rightarrow \quad x = \sqrt{u} \\
&\Rightarrow \quad \mathrm{d}x = \frac{\mathrm{d}u}{2\sqrt{u}} \\
\Rightarrow u \text{ Ranges from } 0 \text{ to } \infty & \quad \text{When } x \text{ ranges from } 0 \text{ to } \infty \\
\Rightarrow \int_0^\infty e^{-x^2} \mathrm{d}x &= \int_0^\infty e^{-u} \frac{\mathrm{d}u}{2\sqrt{u}} \\
&= \frac{1}{2} \int_0^\infty u^{1/2-1} e^{-u} \mathrm{d}u \\
&= \frac{1}{2} \Gamma(\frac{1}{2}) \\
&= \frac{1}{2} \sqrt{\pi}
\end{aligned}$$