The Gamma Distirbution

1 Definition & Motivation

- Often times we would like to model a continuous Random variable that only has a positive support
- In these cases, the Normal distribution is not an appropriate model because a Normally distibruted Random Variable would have a support that included negative numbers
- The Gamma distibution is one of the most commonly used distributions (it actually is a family of distirbutions) for modeling Random variables with a positive support

  Definition 1. *We say that the continuous Random Variable X has a Gamma distribution with parameters α and β when X has the PDF*

$$f_X(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}} & 0 < x \\ 0 & else \end{cases}$$

  Where $0 < \alpha$, $0 < \beta$. We denote this $X \sim \Gamma(\alpha, \beta)$ (Note the support of X is $S = (0, \infty)$).

2 Verify, Mean, Variance, & MGF

Let $X \sim \Gamma(\alpha, \beta)$

1. Verify

   a) $f_X(x) \geq 0 \forall x \in S$

   Proof. We see that $\frac{1}{\Gamma(\alpha)\beta^\alpha} > 0$ by definition and $e^{-\frac{x}{\beta}} > 0$ since $e > 0$. Since $f_X(x)$ is the product of these two positve terms when $x > 0$ and $f_X(x) = 0$ other wise, we see that

$$f_X(x) \geq 0$$

$\square$

   b) $\int_{-\infty}^{\infty} f_X(x) \mathrm{d}x = 1$

Proof.

$$\begin{aligned}
\int_{-\infty}^{\infty} f_X(x)\mathrm{d}x &= \int_{-\infty}^{0} f_X(x)\mathrm{d}x + \int_{0}^{\infty} f_X(x)\mathrm{d}x \\
&= 0 + \int_{0}^{\infty} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}}\mathrm{d}x \\
&= \int_{0}^{\infty} \frac{1}{\Gamma(\alpha)} \frac{1}{\beta} (\frac{x}{\beta})^{\alpha-1} e^{-\frac{x}{\beta}}\mathrm{d}x \\
\text{Integration by substitution} & \\
u &= \frac{x}{\beta} \\
\Rightarrow \mathrm{d}u &= \frac{\mathrm{d}x}{\beta} \\
\Rightarrow u \text{ Ranges from 0 to } \infty & \quad \text{When } x \text{ ranges from 0 to } \infty \\
\Rightarrow \int_{0}^{\infty} \frac{1}{\Gamma(\alpha)} \frac{1}{\beta} (\frac{x}{\beta})^{\alpha-1} e^{-\frac{x}{\beta}}\mathrm{d}x &= \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} (u)^{\alpha-1} e^{-u}\mathrm{d}u \\
&= \frac{1}{\Gamma(\alpha)}\Gamma(\alpha) \\
&= 1
\end{aligned}$$

□

2. Mean

$$E[X] = \alpha\beta$$

Proof.

$$\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} x f_X(x)\mathrm{d}x \\
&= \int_{-\infty}^{0} x f_X(x)\mathrm{d}x + \int_{0}^{\infty} x f_X(x)\mathrm{d}x \\
&= 0 + \int_{0}^{\infty} x \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}}\mathrm{d}x \\
&= 0 + \int_{0}^{\infty} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha} e^{-\frac{x}{\beta}}\mathrm{d}x \\
&= \int_{0}^{\infty} \frac{1}{\Gamma(\alpha)} (\frac{x}{\beta})^{\alpha} e^{-\frac{x}{\beta}}\mathrm{d}x \\
\text{Integration by substitution} & \\
u &= \frac{x}{\beta}
\end{aligned}$$

$$
\begin{aligned}
\Rightarrow \mathrm{d}u &= \frac{\mathrm{d}x}{\beta} \\
\Rightarrow \beta \mathrm{d}u &= \mathrm{d}x \\
\Rightarrow u \text{ Ranges from 0 to } \infty & \quad \text{When } x \text{ ranges from 0 to } \infty \\
\Rightarrow \int_0^\infty \frac{1}{\Gamma(\alpha)}(\frac{x}{\beta})^\alpha e^{-\frac{x}{\beta}}\mathrm{d}x &= \frac{1}{\Gamma(\alpha)}\int_0^\infty (u)^\alpha e^{-u}\beta \mathrm{d}u \\
&= \frac{\beta}{\Gamma(\alpha)}\int_0^\infty (u)^{(\alpha+1)-1}e^{-u}\mathrm{d}u \\
&= \frac{\beta}{\Gamma(\alpha)}\Gamma(\alpha+1) \\
&= \frac{\beta}{\Gamma(\alpha)}\alpha\Gamma(\alpha) \\
&= \alpha\beta
\end{aligned}
$$

□

3. Variance

$$V[X] = \alpha\beta^2$$

Proof. Proof left As HW □

4. MGF

$$M_X(t) = \frac{1}{(1-\beta t)^\alpha} \text{ for } t < \frac{1}{\beta}$$

Proof. Proof left as extra credit □

3 Distributions within the Gamma Family of Distbutions

- Within the Gamma Family of distirbutions, there are sub-families, which are basically special Gamma distributions
- The first special kind of Gamma distribution is the Exponential distirbution:

  Definition 2. *We say that the continuous Random Variable X has an Exponential distribution with the parameter δ when X has the PDF*

  $$f_X(x) = \begin{cases} \frac{1}{\delta}e^{-\frac{x}{\delta}} & 0 < x \\ 0 & else \end{cases}$$

  Where $0 < \delta$. We denote this $X \sim Exp(\delta)$ (Note the support of X is $S = (0, \infty)$).

- Note, when $X \sim Exp(\delta)$, it is also true that $X \sim \Gamma(\alpha = 1, \beta = \delta)$

- The Second Kind of Gamma distibution is the χ^2 (pronounced "Kai-squared" [but spelled chi-squared]) distirbution:

  Definition 3. *We say that the continuous Random Variable X has a χ^2 distribution with the parameter n when X has the PDF*

  $$f_X(x) = \begin{cases} \frac{1}{\Gamma(n/2)2^{(n/2)}} x^{(n/2)-1} e^{-\frac{x}{2}} & 0 < x \\ 0 & \text{else} \end{cases}$$

  Where $0 < n$ (typically an integer value). We denote this $X \sim \chi^2_n$ (Note the support of X is $S = (0, \infty)$).

- Note, when $X \sim \chi^2_n$, it is also true that $X \sim \Gamma(\alpha = \frac{n}{2}, \beta = 2)$

- The parameter n is also refered to as the degrees of freedom of the distribution (i.e. it is a Chi-squared sitribution with n degreees of freedom)

- This name for the n parameter comes from the method bywhich we typically construct a Chi-squared random variable