# Homework 8

**Solutions!**

1. Let $X \sim \Gamma(\alpha, \beta)$. Derive the variance of $X$ without using the Moment Generating Function of $X$
*Solution:*

*Proof.*

$$
\begin{aligned}
V[X] &= E[X^2] - E^2[X] \\
&= E[X^2] - (\alpha\beta)^2 \\
E[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x)\mathrm{d}x \\
&= \int_{-\infty}^{0} x^2 f_X(x)\mathrm{d}x + \int_{0}^{\infty} x^2 f_X(x)\mathrm{d}x \\
&= 0 + \int_0^\infty x^2 \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}}\mathrm{d}x \\
&= 0 + \int_0^\infty x \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha} e^{-\frac{x}{\beta}}\mathrm{d}x \\
&= \int_0^\infty x \frac{1}{\Gamma(\alpha)} (\frac{x}{\beta})^{\alpha} e^{-\frac{x}{\beta}}\mathrm{d}x
\end{aligned}
$$

Integration by substitution

$$
\begin{aligned}
u = \frac{x}{\beta} &\Rightarrow x = \beta u \\
\Rightarrow \mathrm{d}u &= \frac{\mathrm{d}x}{\beta} \\
\Rightarrow \beta\mathrm{d}u &= \mathrm{d}x \\
\Rightarrow u \text{ Ranges from 0 to } \infty & \quad \text{When } x \text{ ranges from 0 to } \infty \\
\Rightarrow \int_0^\infty x \frac{1}{\Gamma(\alpha)} (\frac{x}{\beta})^{\alpha} e^{-\frac{x}{\beta}}\mathrm{d}x &= \frac{1}{\Gamma(\alpha)} \int_0^\infty (\beta u)(u)^\alpha e^{-u} \beta \mathrm{d}u \\
&= \frac{\beta^2}{\Gamma(\alpha)} \int_0^\infty (u)^{\alpha+1} e^{-u}\mathrm{d}u \\
&= \frac{\beta^2}{\Gamma(\alpha)} \int_0^\infty (u)^{(\alpha+2)-1} e^{-u}\mathrm{d}u
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\beta^2}{\Gamma(\alpha)}\Gamma(\alpha+2) \\
&= \frac{\beta^2}{\Gamma(\alpha)}(\alpha+1)\Gamma(\alpha+1) \\
&= \frac{\beta^2}{\Gamma(\alpha)}(\alpha+1)(\alpha)\Gamma(\alpha) \\
&= \alpha^2\beta^2 + \alpha\beta^2 \\
\Rightarrow V[X] &= E[X^2] - E^2[X] \\
&= \alpha^2\beta^2 + \alpha\beta^2 - (\alpha\beta)^2 \\
&= \alpha\beta^2
\end{aligned}
$$

□

2. Let $X \sim Beta(\alpha, \beta)$. Derive the variance of $X$.
   *Solution:*

   *Proof.*

$$
\begin{aligned}
V[X] &= E[X^2] - E^2[X] \\
&= E[X^2] - \left(\frac{\alpha}{\alpha+\beta}\right)^2 \\
E[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x)\mathrm{d}x \\
&= \int_{-\infty}^{0} x f_X(x)\mathrm{d}x + \int_0^1 x^2 f_X(x)\mathrm{d}x + \int_1^{\infty} x f_X(x)\mathrm{d}x \\
&= 0 + \int_0^1 x^2 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\mathrm{d}x + 0 \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{\alpha+1}(1-x)^{\beta-1}\mathrm{d}x \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{(\alpha+2)-1}(1-x)^{\beta-1}\mathrm{d}x \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} B(\alpha+2,\beta) \leftarrow \text{Definition of Beta function} \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)} \leftarrow \text{Beta function Theorem} \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{(\alpha+1)\Gamma(\alpha+1)\Gamma(\beta)}{(\alpha+\beta+1)\Gamma(\alpha+\beta+1)} \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{(\alpha+1)\alpha\Gamma(\alpha)\Gamma(\beta)}{(\alpha+\beta+1)(\alpha+\beta)\Gamma(\alpha+\beta)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)} \\
&= \frac{\alpha^2+\alpha}{(\alpha+\beta)(\alpha+\beta+1)} \\
\Rightarrow V[X] &= E[X^2]-E^2[X] \\
&= \frac{\alpha^2+\alpha}{(\alpha+\beta)(\alpha+\beta+1)} - \left(\frac{\alpha}{\alpha+\beta}\right)^2 \\
&= \frac{(\alpha^2+\alpha)(\alpha+\beta)}{(\alpha+\beta)^2(\alpha+\beta+1)} - \frac{\alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)} \\
&= \frac{(\alpha^2+\alpha)(\alpha+\beta)-\alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)} \\
&= \frac{\alpha^3+\alpha^2+\alpha^2\beta+\alpha\beta-\alpha^3-\alpha^2\beta-\alpha^2}{(\alpha+\beta)^2(\alpha+\beta+1)} \\
&= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}
\end{aligned}
$$

□

3. Let $X \sim Exp(\delta)$. Show tha the CDF of $x$, $F_X(x)$ is

$$
F_X(x) = \begin{cases} 1-e^{-\frac{x}{\delta}} & 0<x \\ 0 & \text{else} \end{cases}
$$

*Solution:*

*Proof.*

$$
\begin{aligned}
F_X(x) &= \int_{-\infty}^{x} f_X(t)\mathrm{d}t \\
&= \begin{cases} \int_{-\infty}^{x} 0\mathrm{d}t & x<0 \\ \int_{-\infty}^{0} 0\mathrm{d}t + \int_0^x \frac{1}{\delta}e^{-\frac{t}{\delta}}\mathrm{d}t & 0<x \end{cases} \\
&= \begin{cases} 0 & x<0 \\ 0 + -e^{-\frac{t}{\delta}}|_0^x & 0<x \end{cases} \\
&= \begin{cases} 0 & x<0 \\ -e^{-\frac{x}{\delta}} - (-e^{-\frac{0}{\delta}}) & 0<x \end{cases} \\
&= \begin{cases} 0 & x<0 \\ 1-e^{-\frac{x}{\delta}} & 0<x \end{cases} \\
&= \begin{cases} 1-e^{-\frac{x}{\delta}} & 0<x \\ 0 & \text{else} \end{cases}
\end{aligned}
$$

□

4. Let $X \sim \Gamma(\alpha, \beta)$ and let $r$ be a real constant such that $r > -\alpha$. Prove that

$$E[X^r] = \frac{\beta^r \Gamma(\alpha + r)}{\Gamma(\alpha)}$$

*Solution:*

*Proof.*

$$\begin{aligned}
E[X^r] &= \int_{-\infty}^{\infty} x^r f_X(x) \mathrm{d}x \\
&= \int_0^{\infty} x^r \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}} \mathrm{d}x \\
&= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^{\infty} x^{\alpha+r-1} e^{-\frac{x}{\beta}} \mathrm{d}x \\
&= \frac{1}{\beta^\alpha} \frac{\Gamma(\alpha+r)\beta^{\alpha+r}}{\Gamma(\alpha+r)\beta^{\alpha+r}} \int_0^{\infty} x^{\alpha+r-1} e^{-\frac{x}{\beta}} \mathrm{d}x \\
&= \frac{\Gamma(\alpha+r)\beta^{\alpha+r}}{\Gamma(\alpha)\beta^\alpha} \int_0^{\infty} \frac{1}{\Gamma(\alpha+r)\beta^{\alpha+r}} x^{\alpha+r-1} e^{-\frac{x}{\beta}} \mathrm{d}x \\
& \quad \text{This is the (valid) pdf of } X \sim \Gamma(\alpha+r, \beta) \text{ since } r > -\alpha \uparrow \\
&= \frac{\Gamma(\alpha+r)\beta^{\alpha+r}}{\Gamma(\alpha)\beta^\alpha} \\
&= \frac{\Gamma(\alpha+r)\beta^r}{\Gamma(\alpha)}
\end{aligned}$$

□