Transformations

1 Motivation

- Our last section of the course Deals with the transformation of Random Variables
- A situation is called a transformation of a random variable when we have a random variable, X, with a given distribution (in the real world this random variable would be some sort of measurement) and then we apply some function h to that random variable.
- What we are concerened with is what is the distribution of the new random variable $h(X)$
- There are multiple methods that we have to determine what the distribution of $h(X)$ is.
- The rest of this leture will be spent establishing some of those methods

2 Method of Distribution Function

Let X be a R.V. with cdf $F_X(x)$ and let $U = h(X)$. Steps:

1. Find formula for CDF of U based on the distribution of X
2. i.e. Find $F_U(u) = P(U < u) = P(h(X) < u)$
3. Differentiate $F_U(u)$ to find pdf of U (only works in the contounous case)

2.1 Examples

$$f(x) = \begin{cases} 2(1-x) & 0 < x < 1 \\ 0 & else \end{cases}$$

1. Let $U = 2X - 1$. Find pdf of U.

$$\begin{aligned} \rightarrow F_U(u) &= P(U < u) \\ &= P(2X - 1 < u) \\ &= P(X < \frac{u+1}{2}) \\ &= \int_{-\infty}^{\frac{u+1}{2}} f_X(x)\mathrm{d}x \\ &= \begin{cases} 0 & \frac{u+1}{2} \leq 0 \\ \int_0^{\frac{u+1}{2}} 2(1-x)\mathrm{d}x & 0 < \frac{u+1}{2} < 1 \\ 1 & \frac{u+1}{2} \geq 1 \end{cases} \end{aligned}$$

$$
\begin{aligned}
&= \begin{cases} 0 & \frac{u+1}{2} \leq 0 \\ 2\frac{u+1}{2} - (\frac{u+1}{2})^2 & 0 < \frac{u+1}{2} < 1 \\ 1 & \frac{u+1}{2} \geq 1 \end{cases} \\
&= \begin{cases} 0 & u < -1 \\ \frac{u}{2} - \frac{u^2}{4} + \frac{3}{4} & -1 < u < 1 \\ 1 & u \geq 1 \end{cases} \\
\Rightarrow f_U(u) &= \frac{\mathrm{d}}{\mathrm{d}u} F_U(u) \\
&= \begin{cases} \frac{1}{2} - \frac{u}{2} & -1 < u < 1 \\ 0 & else \end{cases}
\end{aligned}
$$

2. Let $U = X^2$. Find the pdf of U

$$
\begin{aligned}
\rightarrow F_U(u) &= P(U < u) \\
&= P(X^2 < u) \\
&= P(-\sqrt{u} < X < \sqrt{u}) \\
&= \int_{-\sqrt{u}}^{\sqrt{u}} f_x(x)\mathrm{d}x \\
&= \begin{cases} 0 & \sqrt{u} \leq 0 \\ \int_0^{\sqrt{u}} 2(1-x)\mathrm{d}x & 0 < \sqrt{u} < 1 \\ 1 & \sqrt{u} \geq 1 \end{cases} \\
&= \begin{cases} 0 & u \leq 0 \\ 2\sqrt{u} - (\sqrt{u})^2 & 0 < u < 1 \\ 1 & u \geq 1 \end{cases} \\
\Rightarrow f_U(u) &= \frac{\mathrm{d}}{\mathrm{d}u} F_U(u) \\
&= \begin{cases} \frac{1}{\sqrt{u}} - 1 & 0 < u < 1 \\ 0 & else \end{cases}
\end{aligned}
$$

3. Let X and Y have the joint pdf:

$$
f(x,y) = \begin{cases} 1 & 0 < x < 1;\ 0 < y < 1 \\ 0 & else \end{cases}
$$

Find the pdf of $U = X + Y$

$$
\begin{aligned}
\rightarrow F_U(u) &= P(U < u) \\
&= P(X + Y < u)
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^u \int_0^{u-x} f(x,y)\mathrm{d}y\mathrm{d}x \\
&= \begin{cases} 0 & u \le 0 \\ \int_0^u \int_0^{u-x} 1\mathrm{d}y\mathrm{d}x & 0 < u \le 1 \\ 1 - \int_{u-1}^1 \int_{u-x}^1 1\mathrm{d}y\mathrm{d}x & 1 \le u < 2 \\ 1 & 2 \le u \end{cases} \\
&= \begin{cases} 0 & u \le 0 \\ \frac{u^2}{2} & 0 < u \le 1 \\ 1 - \frac{1}{2}(2-u)^2 & 1 \le u < 2 \\ 1 & 2 \le u \end{cases} \\
\Rightarrow f_U(u) &= \frac{\mathrm{d}}{\mathrm{d}u} F_U(u) \\
&= \begin{cases} u & 0 < u \le 1 \\ 2-u & 1 \le u < 2 \\ 0 & else \end{cases}
\end{aligned}
$$

3 Method of Transformation (Jacobian)

Let X be a R.V. with pdf $f_X(x)$ and let $U = h(X)$. Steps:

1. Verify that $h(X)$ is either increasing or decreasing over support of X
2. Find $h^{-1}(u)$ such that $h(X) = U \Rightarrow h^{-1}(U) = X$
3. Find derivative $\frac{\mathrm{d}}{\mathrm{d}u} h^{-1}(u)$
4. $f_U(u) = f_X(h^{-1}(u)) |\frac{\mathrm{d}}{\mathrm{d}u} h^{-1}(u)|$

3.1 Examples

$$
f(x) = \begin{cases} 2(1-x) & 0 < x < 1 \\ 0 & else \end{cases}
$$

1. Let $U = 2X - 1$. Find pdf of U.
   i. $h(X) = 2X - 1$ is increasing in X
   ii. $h(X) = 2X - 1 \Rightarrow h^{-1}(U) = \frac{U+1}{2}$
   iii. $\frac{\mathrm{d}}{\mathrm{d}u} h^{-1}(u) = \frac{1}{2}$
   iv.

$$
\begin{aligned}
f_U(u) &= f_X(h^{-1}(u)) |\frac{\mathrm{d}}{\mathrm{d}u} h^{-1}(u)| \\
&= \begin{cases} 2(1 - \frac{u+1}{2}) |\frac{1}{2}| & 0 < \frac{u+1}{2} < 1 \\ 0 & else \end{cases}
\end{aligned}
$$

$$= \begin{cases} (\frac{1}{2} - \frac{u}{2}) & -1 < u < 1 \\ 0 & else \end{cases}$$

2. Let $U = X^2$. Find the pdf of U
   i. $h(X) = X^2$ is increasing in X. Note: This is because X only ranges from 0 to 1.
   ii. $h(X) = X^2 \Rightarrow h^{-1}(U) = \sqrt{U}$. Again this because X does not take on negative values.
   iii. $\frac{\mathrm{d}}{\mathrm{d}u} h^{-1}(u) = \frac{1}{2\sqrt{u}}$
   iv.

$$\begin{aligned} f_U(u) &= f_X(h^{-1}(u))|\frac{\mathrm{d}}{\mathrm{d}u} h^{-1}(u)| \\ &= \begin{cases} 2(1-\sqrt{u})|\frac{1}{2\sqrt{u}}| & 0 < \sqrt{u} < 1 \\ 0 & else \end{cases} \\ &= \begin{cases} \frac{1}{\sqrt{u}} - 1 & 0 < u < 1 \\ 0 & else \end{cases} \end{aligned}$$

4 Method of MGF

Let X be a R.V. with MGF $M_X(t)$ and let $U = h(X)$. Steps:

1. Note:

$$\begin{aligned} M_U(t) &= E[e^{Ut}] \\ &= E[e^{h(X)t}] \end{aligned}$$

2. Identify the form of $M_U(t)$ as that of a known MGF
3. Based on the MGF, identify the pdf of U

4.1 Examples

1. Let $X \sim \mathrm{N}(0, 1)$. Let $U = 2X$. Find the pdf of U

$$\begin{aligned} M_U(t) &= E[e^{Ut}] \\ &= E[e^{2Xt}] \\ &= M_X(2t) \\ &= e^{\frac{(2t)^2}{2}} \\ &= e^{\frac{2^2 t^2}{2}} \\ &\Rightarrow U \sim \mathrm{N}(0, 4) \end{aligned}$$

2. Let $X, Y \sim \mathrm{N}(0,1)$ and let X and Y be independent. Find the pdf of $U = X + Y$

$$\begin{aligned}
M_U(t) &= E[e^{Ut}] \\
&= E[e^{(X+Y)t}] \\
&= E[e^{Xt+Yt}] \\
&= E[e^X e^{Yt}] \\
&= E[e^X]E[e^{Yt}] \\
&= M_X(t)M_Y(t) \\
&= e^{\frac{t^2}{2}} e^{\frac{t^2}{2}} \\
&= e^{\frac{2t^2}{2}} \\
&\Rightarrow U \sim \mathrm{N}(0,2)
\end{aligned}$$

5 Exercises

Let X have the following pdf

$$f_X(x) = \begin{cases} \frac{1}{2} & -1 < x < 1 \\ 0 & else \end{cases}$$

1. Find the pdf of $U = |X|$
2. Find the pdf of $W = X^3$
3. Let $X, Y \sim \mathrm{Exp}(1)$ and let X and Y be independent. find the pdf of $U = X + Y$

6 Solutions

1.

$$\begin{aligned}
\rightarrow F_U(u) &= P(U < u) \\
&= P(|X| < u) \\
&= P(-u < X < u) \\
&= \int_{-u}^{u} f_x(x)\mathrm{d}x \\
&= \begin{cases} 0 & u \leq 0 \\ \int_{-u}^{u} \frac{1}{2}\mathrm{d}x & 0 < u < 1 \\ 1 & u \geq 1 \end{cases} \\
&= \begin{cases} 0 & u \leq 0 \\ u & 0 < u < 1 \\ 1 & u \geq 1 \end{cases} \\
\Rightarrow f_U(u) &= \frac{\mathrm{d}}{\mathrm{d}u} F_U(u)
\end{aligned}$$

$$= \begin{cases} 1 & 0 < u < 1 \\ 0 & else \end{cases}$$

2. i. $h(X) = X^3$ is increasing in X. Note: This is because X^3 is an odd function

   ii. $h(X) = X^3 \Rightarrow h^{-1}(U) = \sqrt[3]{U}$.

   iii. $\frac{\mathrm{d}}{\mathrm{d}u} h^{-1}(u) = \frac{u^{-\frac{2}{3}}}{3}$

   iv.

$$\begin{aligned} f_U(u) &= f_X(h^{-1}(u))\left|\frac{\mathrm{d}}{\mathrm{d}u} h^{-1}(u)\right| \\ &= \begin{cases} \frac{1}{2}\left|\frac{u^{-\frac{2}{3}}}{3}\right| & -1 < \sqrt[3]{u} < 1 \\ 0 & else \end{cases} \\ &= \begin{cases} \frac{u^{-\frac{2}{3}}}{6} & -1 < u < 1 \\ 0 & else \end{cases} \end{aligned}$$

3.

$$\begin{aligned} M_U(t) &= E[e^{Ut}] \\ &= E[e^{(X+Y)t}] \\ &= E[e^{Xt+Yt}] \\ &= E[e^{X} e^{Yt}] \\ &= E[e^{X}]E[e^{Yt}] \\ &= M_X(t)M_Y(t) \\ &= \frac{1}{1-t}\frac{1}{1-t} \text{ When } |t| < 1 \\ &= \frac{1}{(1-t)^2} \text{ When } |t| < 1 \\ &\Rightarrow U \sim \Gamma(2,1) \end{aligned}$$