1 Probability Experiments

- Experiments
  - A *probability experiement* is an activity that involves chance that leads to results, that can be repeated
- Sample Points
  - A *Sample point* is a possible result of a single trial, or repetition, of an experiment
- Event
  - An *event* is one or more possible outcomes of an experiment (a set of sample points)
    * A simple event is an event with a single sample point
    * A compound event is an event of more than one sample point
  - We denote the probability of an event A as $P(A)$
  - Note that an event is always a subset of the sample space
- Sample Space
  - A *Sample Space* of an experiement is the set of all possible sample points for an experiment
  - Note, by the definintion of an event, this makes a sample space a special kind of event.
- Examples:
  - Consider the probability experiment where we flip a coin
    * The possible outcomes of this experiment are that the coin lands heads up, or it lands tails up
    * Thus, our sample points are that the coin lands heads up (which we represent with H) and that the coin lands talis up (which we represent with a T)
    * Therefore, the sample space is $S = \{H, T\}$
  - Consider the probability experiement that we roll a six-sided die
    * The possible outcomes of this experiment are that the die rolls with a a number (1,2,3,4,5, or 6) facing up
    * So, our sample points are that the die has 1,2,3,4,5, or 6 showing. we will represent these sample points with the numbers 1,2,3,4,5, and 6.

    * Thus, our sample space is $S = \{1, 2, 3, 4, 5, 6\}$
    * A simple event would be that the die has the number 1 facing up, and would be represented as $\{1\}$
    * A compund event would be that the die has an even number facing up, and would be represented as $\{2, 4, 6\}$

- Compliments of events
  - The compliment of an event [A] (or the negation of the description of an event) is the event in which A does not occur.
  - For example, in our die rolling experiment, the compliment of the event that we roll an even number would be the event that we DO NOT roll an even number.
  - Since events are sets, the compliment of an event (A) will be the event (B) with every sample point in the sample space that isn't in the original event (A). i.e. $B = A^C$
  - The compliment of the event that we roll an even numer would be $\{1, 3, 5\}$
- Union of events
  - The union of two events is the event in which either of those events happen
  - For example the union of the event that we roll an even number and the event that we roll a number elss than 3, would be the event that we roll a number that is even or less than 3
  - Since events are sets, we carry out this union through a set union
  - Example:
    * The event that we roll and even number is $A = \{2, 4, 6\}$, and the event that we roll a number less than 3 is $B = \{1, 2\}$
    * So, the event that we roll a number that is either even or less than 3 would be $A \cup B = \{1, 2, 4, 6\}$
- Intersection of events
  - The intersection of two events is the event in which both of those events happen
  - For example the intersection of the event that we roll an even number and the event that we roll a number elss than 3, would be the event that we roll a number that is even and less than 3
  - Since events are sets, we carry out this intersection through a set intersection
  - Example:
    * The event that we roll and even number is $A = \{2, 4, 6\}$, and the event that we roll a number less than 3 is $B = \{1, 2\}$
    * So, the event that we roll a number that is both even and less than 3 would be $A \cap B = \{2\}$

2 Probability

- Probability experiments have the following 4 Axioms:

  1. Has a set, S, of distinct possible outcomes (sample points)(We defined that as the sample space above)
  2. The probaility of the event that is the sample space is 1 (i.e. one of the possible sample points must occur when the experiement happens)
  3. For each subset A of S, the probability of that event A occurs is greatehr than or equal to 0.
     - We write "the probability of that event A occurs is greatehr than or equal to 0" as $P(A) \geq 0$
  4. If $A_1, A_2, A_3, ...$ are pairwise disjoint subsets of S for some whole number n, then $P(A_1 \cup A_2 ...) = P(A_1) + P(A_2) + ...$

- *Note:* For the events $A_1, A_2, A_3, ...$ to be pairwise disjoint then any pair you select from the n set must be disjoint.

- Formally we say that $A_1, A_2, A_3, ...$ are pairwise disjoint if and only if $A_i \cap A_j = \varnothing \forall i, j$ such that $i \neq j$. Remember the $\forall$ symbol is read as "For all"

- We calculate the probability of an event In the following way:
  - $P(A)$ (read the probability of event A) is the sum of the probabilities of the sample points contained within set A

- Examples:
  - Experiment: flip a fair coin
    * Sample space: ? ({H, T}) (remember, H stands for the coin landing heads uo, and T stands for the coin landing tails up)
    * Since we say that the coin is *fair*, this means that the probability of every sample point is the same
    * So, how do we find $P(\{H\}) =?$
    * Since the probability of every samplepoint is the same we know that $P(\{H\}) = P(\{T\})$
    * We also know tat the probabilities of all of the sample points must sum to 1 from rules 2 and 4.
    * That is to say, we know that $P(S) = 1$ and we know that Since $S = \{H, T\}$, $P(S) = P(\{H\}) + P(\{T\})$
    * Since we know that $P(\{H\}) = P(\{T\})$, we can derive the $P(\{H\}$ with the following steps:

$$
\begin{aligned}
P(S) &= 1 \\
\Rightarrow P(\{H\}) + P(\{T\}) &= 1 \leftarrow \text{Since } S = \{H, T\} \\
\Rightarrow 2P(\{H\} &= 1 \leftarrow \text{Since } P(\{H\}) = P(\{T\}) \\
\Rightarrow P(\{H\}) &= \frac{1}{2}
\end{aligned}
$$

- Experiment: roll a fair die
    - Sample Space: ? ({1, 2, 3, 4, 5, 6})
    - $P(\{1\}) = ?$ (1/6)
    - Let A be the event that we roll an even number. $P(A) = ?$ (3/6)
    - Let B be the event that we roll a number less than 3. $P(B) = ?$ (2/6)
    - What is the probbaility that we roll a number that is even or less than 3? (4/6)

3 Probability Rules

- General Additive Rule
    - For any two events A and B, $P(A \cup B) = P(A) + P(B) – P(A \cap B)$
- Mutually Exclusive
    - Two events A and B are called mutually exclusive if $A \cap B = \varnothing$, the empty set
    - In other words, mutually exclusive is a way of saying that events are disjoint
    - This implies that $P(A \cap B) = P(\varnothing) = 0$
- Independence
    - Two events are said to be independent if $P(A \cap B) = P(A) \times P(B)$
    - Given more than two events, for example the n events $A_1, A_2, .., A_n$, then these events are said ot be independent if

$$P(\cap_{i \in \tau} A_i) = \prod_{i \in \tau} P(A_i), \forall \tau \subset \{1, 2, 3..., n\}$$

- Conditional probability
    - We say that the probability of event A occurs given that event B has already occurred is the conditional probability of event A given event B
    - This probability is denoted $P(A|B)$
    - This probability is defined to be $P(A|B) = P(A \cap B)/P(B)$

4 Resulting Probability Properties

- General additive rule + mutually exclusive
    - If two events are mutually exclusive then $P(A \cup B) = P(A) + P(B)$
- Conditional probability + independence

– If two events are independent then

$$\begin{aligned} P(A|B) &= P(AUB)/P(B) \\ &= (P(A) * P(B))/P(B) \\ &= P(A) \end{aligned}$$

5 Exercises

1. Let A and B be events in the sample space S. Prove that

$$P(A \cup B) = P(A) + P(B \cap A^C)$$

Solution:

Proof. First, we will show $A \cup B = A \cup (B \cap A^C)$. There are two approaches to showing this equality; first we will prove the equality by demonstrating that the two are mutual subsets of each other.

This means we want to show that

(a) $A \cup B \subset A \cup (B \cap A^C)$, and

(b) $A \cup B \supset A \cup (B \cap A^C)$

First, we show that $A \cup B \subset A \cup (B \cap A^C)$.

Let $a \in A \cup B$. This implies $a \in A$ or $a \in B$.

$$\begin{aligned} \text{Case 1: } a \in A & \\ & \text{If } a \in A \text{ then } a \in A \cup (B \cap A^C) \\ \text{Case 2: } a \in B & \\ & \text{If } a \in B \text{ then either} \\ & a \in A \text{ or } a \in A^C \\ \text{Case 2(a): } a \in A & \\ & \text{If } a \in A \text{ then} a \in A \cup (B \cap A^C) \\ \text{Case 2(b): } a \in A^C & \\ & \text{If } a \in A^C \text{ then } a \in (B \cap A^C) \\ \Longrightarrow & a \in A \cup (B \cap A^C). \end{aligned}$$

Next, we show that $A \cup B \supset A \cup (B \cap A^C)$.

Let $a \in A \cup (B \cap A^C)$. This implies $a \in A$ or $a \in (B \cap A^C)$.

$$\begin{aligned}
\text{Case 1: } & a \in A \\
& \text{If } a \in A \text{ then } a \in A \cup B \\
\text{Case 2: } & a \in (B \cap A^C \\
& \text{If } a \in (B \cap A^C) \text{ then } a \in B \\
& \implies a\ inA \cup B.
\end{aligned}$$

A second approach to proving the equality is to directly apply the distributive law of sets:

$$\begin{aligned}
A \cup (B \cap A^C) &= (A \cup B) \cap (A \cup A^C) \leftarrow \text{By the Distributive Law} \\
&= (A \cup B) \cap S \\
&= A \cup B.
\end{aligned}$$

Now that we have shown that $A \cup B = A \cup (B \cap A^C)$, we see that

$$\begin{aligned}
P(A \cup B) &= P(A \cup (B \cap A^C)) \\
&= P(A) + P(B \cap A^C) - P(A \cap B \cap A^C) \leftarrow \text{Gen. Add. Rule} \\
&= P(A) + P(B \cap A^C) \qquad (1) \\
&\uparrow \text{Since } A \cap B \cap A^C = B \cap (A \cap A^C) = \varnothing
\end{aligned}$$

□

2. Prove the general Additive Rule.

Proof. We begin by noting that with sample space S, we can change the representation of A, that is

$$\begin{aligned}
A &= A \cap S \leftarrow \text{Since } A \subset S \\
&= A \cap (B \cup B^C) \leftarrow \text{By def of set compliment} \qquad (2) \\
&= (A \cap B) \cup (A \cap B^C). \leftarrow \text{Distributive Law}
\end{aligned}$$

Furthermore, we see that

$$\begin{aligned}
(A \cap B) \cap (A \cap B^C) &= (A \cap A) \cap (B \cap B^C) \leftarrow \text{Associative and commutative laws} \\
&= A \cap \varnothing \leftarrow \text{By def of set compliment} \\
& \varnothing.
\end{aligned}$$

Therefore, sets $A \cap B$ and $A \cap B^C$ are disjoint, or mutually exclusive. This means that when we apply the probability function P to both sides of Equation (2) we get

$$
\begin{aligned}
P(A) &= P((A \cap B) \cup (A \cap B^C)) \\
&= P(A \cap B) + P(A \cap B^C). \qquad (3) \\
&\quad \uparrow \text{Since the events are mutually exclusive, we apply axiom 4}
\end{aligned}
$$

Now, we can consider the right hand side of the desired equation:

$$
\begin{aligned}
P(A) + P(B) - P(A \cap B) &= P(A \cap B) + P(A \cap B^C) + P(B) - P(A \cap B) \leftarrow \text{From Equation (3)} \\
&= P(A \cap B^C) + P(B) \\
&= P(A \cup B) \leftarrow \text{ Shown in previous problem; see Equation(1)}
\end{aligned}
$$

□

3. Let A, B, and C be events. Prove

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Proof. We begin by simplifying the problem temporarily by letting $D = B \cup C$. Then we see

$$
\begin{aligned}
P(A \cup B \cup C) &= P(A \cup D) \\
&= P(A) + P(D) - P(A \cap D) \leftarrow \text{Gen. Add. Rule} \\
&= P(A) + P(B \cup C) - P(A \cap (B \cup C)) \leftarrow \text{Substituing back in} \\
&= P(A) + P(B \cup C) - P((A \cap B) \cup (A \cap C)) \leftarrow \text{Distributive Law} \\
&= P(A) + [P(B) + P(C) - P(B \cap C)] - [P(A \cap B) + P(A \cap C) - P((A \cap B) \cap (A \cap C))] \\
&\qquad \uparrow \text{Gen. Add. Rule} \\
&= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C) \\
&\qquad \uparrow \text{Since } (A \cap B) \cap (A \cap C) = (A \cap A) \cap (B \cap C)
\end{aligned}
$$

□