1 Geometric Distribution

- Imagine we are again working with a coin that has a probability of landing heads up of p where $0 < p < 1$
- Suppose that instead of flipping the coin a certain number of times like we did to generate the Binomial Distribution, we flip the coin until it lands heads up
- What would the sample space of this experiment be? Lets consider thet sample points:

e_1 :	(H)	Observe head on the first flip
e_2 :	(T, H)	Observe head on the second flip
e_3 :	(T, T, H)	Observe head on the third flip
$\vdots$		
e_n :	$(T, T, ..., H)$	Observe head on the n^{th} flip
$\vdots$		

- This would make our sample space

$$S = \{e_1, e_2, ..., e_n, ...\}$$

- How do we calculate the probability of these sample points?
- We will assume that since we are flipping the same coin repeatedly, the probability of the coin landing heads on any given flip will be p (and the probability of tails will be $1 - p$) and that each coin flip is independent
- This means that :

$$\begin{aligned}
P(\{e_1\}) &= P(\{(H)\}) \\
&= p \\
P(\{e_2\}) &= P(\{(T, H)\}) \\
&= P(T)P(H) \leftarrow \text{Due to independence} \\
&= (1-p)p \\
P(\{e_3\}) &= P(\{(T, T, H)\}) \\
&= P(T)P(T)P(H) \leftarrow \text{Due to independence} \\
&= (1-p)^2 p \\
&\vdots \\
P(\{e_n\}) &= P(\{(T, T, ..., H)\})
\end{aligned}$$

$$
\begin{aligned}
&= P(T)P(T)...P(H) \leftarrow \text{Due to independence} \\
&= (1-p)^{n-1}p \\
&\vdots
\end{aligned}
$$

- Now lets turn this p[robability experiment into a Random Variable. Let $X = \#$ of flips we must make in order to observe our first heads.
- We see that it can possibly take $1, 2, 3, ...$ flips before we observe our first heads, so our support for X will be $\{1, 2, 3, ...\}$
- We see that each simple event $X = x$ for some value x in the support directly corresponds to a sample point in the original experiment:

$$
\begin{aligned}
X = 1 &\rightarrow e_1 \\
X = 2 &\rightarrow e_2 \\
X = 3 &\rightarrow e_3 \\
&\vdots \\
X = n &\rightarrow e_n \\
&\vdots
\end{aligned}
$$

- This means that our probabilities for these events will be:

$$
\begin{aligned}
P(X = 1) &= p(\{e_1\}) \\
&= p \\
P(X = 2) &= p(\{e_2\}) \\
&= (1-p)p \\
P(X = 3) &= p(\{e_3\}) \\
&= (1-p)^2 p \\
&\vdots \\
P(X = n) &= p(\{e_n\}) \\
&= (1-p)^{n-1}p \\
&\vdots
\end{aligned}
$$

- This allows us to write our pdf as

$$p_X(x) = (1-p)^{x-1}p$$

2 Negative Biomial

- Suppose We now extend the probability experiment
- Instead of flipping until we observe 1 head, we will now flip repeatedly until we observe the r^{th} head, where r is a positive integer
- Again, we lets $X = \#$ of flips in order to observe the r^{th} head
- How do we findthe probability $P(X = x)$?
- We can think of the event $X = x$ as the union of of the event that we observe $r-1$ heads in $x-1$ flip and the event that we observe a head on the x^{th} flip
- This means that we can break dow nthe probability like this:

$$\begin{aligned} P(X=x) &= P(\text{observe } r^{th} \text{ head on } x^{th} \text{ flip}) \\ &= P(\text{observe } r-1 \text{ heads in } x-1 \text{ flips} \cap \text{observe head on } x^{th} \text{ flip}) \\ &= P(\text{observe } r-1 \text{ heads in } x-1 \text{ flips})P(\text{observe head on } x^{th} \text{ flip}) \leftarrow \text{Since the events} \\ &= \binom{x-1}{r-1} p^{r-1}(1-p)^{(x-1)-(r-1)}p \leftarrow \text{First probability comes from our analysis of bino} \\ &= \binom{x-1}{r-1} p^{r}(1-p)^{x-r} \end{aligned}$$

- So, our PDF for X is $p_X(x) = \binom{x-1}{r-1}p^r(1-p)^{x-r}$.
- Formally, we say that if a random variable X has the PDF $p_X(x) = \binom{x-1}{r-1}p^r(1-p)^{x-r}$ where r is a positive integer and $0 < p < 1$, then X has a negative binomial distribution with paramters r and p. Symbolically we write this as $X \sim NBinom(r,p)$

3 Verification, Mean, & Variance

- As we will esatablish the following for these PDFs
  1. Verify that the PDF is valid (i.e., show that it follows the two rules for discrete R.V. PDFs)
  2. Establish the mean of the distribution
  3. Establish the variance of the distribution
- *Note:* If we examine the PDFs of the geometric distribution anf the Negative Binomial Distribution we see that If a Random variable $X \sim Geo(p)$, then we can say that $X \sim NBinom(1,p)$
- This means hat if we can verify the distirbution, the mean, and the variance, for the negative binomial distribution, then we will have also done this for the geometric distribution as well

3.1 Negative Binomial

- Let $X \sim NBinom(r, p)$ here r is a positive integer and $0 < p < 1$
  1. Verification
     a) $\sum_{x \in S} p_X(x) = 1$

     Proof. First, we consider the taylor expansion of the function $(1-w)^{-r}$ where $0 < w < 1$ and r is a positive integer. The expansion is

$$\begin{aligned}
(1-w)^{-r} &= \sum_{i=0}^{\infty} \frac{\mathrm{d}^i}{(\mathrm{d}w)^i}(1-w)^{-r}|_{w=0}\frac{(w-0)^i}{i!} \\
&= \sum_{i=0}^{\infty} \frac{(r+i-1)!}{(r-1)!}\frac{w^i}{i!} \\
&= \sum_{i=0}^{\infty} \frac{(r+i-1)!}{(r-1)!((r+i-1)-(r-1)!}w^i \\
&= \sum_{i=0}^{\infty} \binom{r+i-1}{r-1} w^i
\end{aligned}$$

     Now, lets consider $\sum_{x \in S} p_X(x) = 1$:

$$\begin{aligned}
\sum_{x \in S} p_X(x) &= \sum_{x=r}^{\infty} \binom{x-1}{r-1} p^r (1-p)^{x-r} \\
&= \sum_{i=0}^{\infty} \binom{r+i-1}{r-1} p^r (1-p)^{(r+i)-r} \leftarrow \text{Let } x = r+i \\
&= p^r \sum_{i=0}^{\infty} \binom{r+i-1}{r-1} (1-p)^i \\
&= p^r (1-(1-p)^{-r} \leftarrow \text{Taylor expansion shown above} \\
&= p^r p^{-r} \\
&= 1
\end{aligned}$$

     □

     b) $0 \leq p_X(x) \leq 1 \forall x \in S$

     Proof. Since $0 < p < 1$ we see that $0 < 1-p < 1$. Therefore $0 \leq p^r \leq 1$ and $0 \leq (1-p)^{x-r} \leq 1$ for all positive values of r and x such that $x \geq r$. additionally we see that $0 \leq \binom{x-1}{r-1}$, wherefore $0 \geq P_X(x) \forall x \in S$ since $P_X(x)$ is the product of three non-negative things $\forall x \in S$. Since the sum of these individual values of the PDF is 1 and the individiual terms are all non negative, we can conclude that $P_X(x) \leq 1 \forall x \in S$

$\square$

c) Mean

$$E[X] = \frac{r}{p}$$

Proof. Left as an exercise $\square$

d) Variance

$$V[X] = \frac{r(1-p)}{p^2}$$

Proof. Left as an exercise $\square$

3.2 Exercises

Let $X \sim Geo(p)$. Without referring to the Negative Binomial properties, prove the following:

1. Verify that the PDF is valid (i.e., show that it follows the two rules for discrete R.V. PDFs)

2. Establish the mean of the distribution. (i.e. show that $E[X] = 1/p$)

   Hint: Note that it can be show that $\sum_{i=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}p} - (1-p)^x = \frac{\mathrm{d}}{\mathrm{d}p} \sum_{x=1}^{\infty} -(1-p)^x$ and that $\frac{\mathrm{d}}{\mathrm{d}p} - (1-p)^x = x(1-p)^{x-1}$

3. Establish the variance of the distribution. (i.e. show that $V[X] = \frac{1-p}{p^2}$)

   Hint: Note that it can be show that $\sum_{i=1}^{\infty} \frac{\mathrm{d}^2}{\mathrm{d}p^2}(1-p)^{x+1} = \frac{\mathrm{d}^2}{\mathrm{d}p^2} \sum_{x=1}^{\infty}(1-p)^{x+1}$ and that $\frac{\mathrm{d}^2}{\mathrm{d}p^2}(1-p)^{x+1} = (x+1)x(1-p)^{x-1}$

3.3 Solutions:

1. Verification

   a) $\sum_{x \in S} p_X(x) = 1$

   Proof.

$$\begin{aligned}
\sum_{x \in S} p_X(x) &= \sum_{x=1}^{\infty} p(1-p)^{x-1} \\
&= p \sum_{x=1}^{\infty} (1-p)^{x-1}
\end{aligned}$$

$$
\begin{aligned}
&= p \sum_{x-1=0}^{\infty} (1-p)^{x-1} \leftarrow \text{re-indexing the summation} \\
&= p \sum_{y=0}^{\infty} (1-p)^{y} \leftarrow \text{Let } y = x-1 \text{ to clarify algebra} \\
&= p \frac{1}{1-(1-p)} \leftarrow \text{Sum of a geometric series} \\
&= 1
\end{aligned}
$$

□

b) $0 \leq p_X(x) \leq 1 \forall x \in S$

Proof. Since $0 < p < 1$ we see that $0 < 1-p < 1$ and therefore $0 < (1-p)^{x-1} < 1 \forall x \geq 1$. This in turn implies that $0 < p(1-p)^{x-1} < 1 \forall x \in S$ □

2. Mean

Proof.

$$
\begin{aligned}
E[X] &= \sum_{x \in S} x p_X(x) \\
&= \sum_{x=1}^{\infty} x p (1-p)^{x-1} \\
&= p \sum_{x=1}^{\infty} x (1-p)^{x-1} \\
&= p \sum_{x=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}p} - (1-p)^{x} \leftarrow \text{From hint} \\
&= p \frac{\mathrm{d}}{\mathrm{d}p} \sum_{x=1}^{\infty} -(1-p)^{x} \leftarrow \text{From hint} \\
&= p \frac{\mathrm{d}}{\mathrm{d}p} - \left[(1-p) \frac{1}{1-(1-p)} \right] \leftarrow \text{Sum of a geometric series} \\
&= p \frac{\mathrm{d}}{\mathrm{d}p} [1 - \frac{1}{p}] \\
&= p [\frac{1}{p^2}] \\
&= \frac{1}{p}
\end{aligned}
$$

□

3. Variance

Proof.

$$
\begin{aligned}
V[X] &= E[X^2] - E^2[X] \\
\text{also,} & \\
E[(X+1)X] &= E[X^2] + E[X] \\
\Rightarrow E[X^2] &= E[(X+1)X] - E[X] \\
&= E[(X+1)X] - 1/p \\
E[(X+1)X] &= \sum_{x \in S} (x+1) x p_X(x) \\
&= \sum_{x=1}^{\infty} (x+1) x p (1-p)^{x-1} \\
&= p \sum_{x=1}^{\infty} (x+1) x (1-p)^{x-1} \\
&= p \sum_{x=1}^{\infty} \frac{\mathrm{d}^2}{\mathrm{d}p^2} (1-p)^{x+1} \leftarrow \text{From hint} \\
&= p \frac{\mathrm{d}^2}{\mathrm{d}p^2} \sum_{x=1}^{\infty} (1-p)^{x+1} \leftarrow \text{From hint} \\
&= p \frac{\mathrm{d}^2}{\mathrm{d}p^2} \left[(1-p)^2 \frac{1}{1-(1-p)} \right] \leftarrow \text{Sum of a geometric series} \\
&= p \frac{\mathrm{d}^2}{\mathrm{d}p^2} [\frac{1}{p} - 2 + p] \\
&= p[\frac{2}{p^3}] \\
&= \frac{2}{p^2} \\
\Rightarrow E[X^2] &= \frac{2}{p^2} - \frac{1}{p} \\
&= \frac{2}{p^2} - \frac{p}{p^2} \\
&= \frac{2-p}{p^2} \\
\Rightarrow V[X] &= E[X^2] - E^2[X] \\
&= \frac{2-p}{p^2} - \frac{1}{p^2} \\
&= \frac{1-p}{p^2}
\end{aligned}
$$

□