Multivariate Jacobian Transformations

1 Motivation

- Sometimes we are presented with a transformation that can't be solved with a previous trasnformation.
- The most common issue is that we need to transform more than one random variable at a time but the random variables being transformed are not independent/identical (if they were independent & identical the MGF technique would work well)

2 Outline of Problem & Solution

- Let $X_1, X_2, ...X_n$ are Random variables with a joint pdf $f(x_1, x_2, ...x_n)$ and suppose we have transformations $U_1 = h_1(X_1, X_2, ...X_n), U_2 = h_2(X_1, X_2, ...X_n), ..., U_n = h_n(X_1, X_2, ...X_n)$ such that $h_1, h_2, ...h_n$ form a one-to-one transformation from $X_1, X_2, ..., X_n$ to $U_1, U_2, ..., U_n$.
- Since these are a one-to-one transformation we know that there exists $h_1^{-1}, h_2^{-1}, ..., h_n^{-1}$ that "undoes" the transformations performed by $h_1, h_2, ..., h_n$... i.e. $X_1 = h_1^{-1}(U_1, U_2, ..., U_n)$, $X_2 = h_2^{-1}(U_1, U_2, ..., U_n), ..., X_n = h_n^{-1}(U_1, U_2, ..., U_n)$
- Then the joint distribution of $U_1, U_2, ..., U_n$ is

$$f(h_1^{-1}(U_1, U_2, ..., U_n), h_2^{-1}(U_1, U_2, ..., U_n), ..., h_n^{-1}(U_1, U_2, ..., U_n))|J|$$

Where $J = det \begin{pmatrix} \frac{\mathrm{d}h_1^{-1}}{\mathrm{d}u_1} & \frac{\mathrm{d}h_2^{-1}}{\mathrm{d}u_1} & \cdots & \frac{\mathrm{d}h_n^{-1}}{\mathrm{d}u_1} \\ \frac{\mathrm{d}h_1^{-1}}{\mathrm{d}u_2} & \frac{\mathrm{d}h_2^{-1}}{\mathrm{d}u_2} & \cdots & \frac{\mathrm{d}h_n^{-1}}{\mathrm{d}u_2} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ \frac{\mathrm{d}h_1^{-1}}{\mathrm{d}u_n} & \frac{\mathrm{d}h_2^{-1}}{\mathrm{d}u_n} & \cdots & \frac{\mathrm{d}h_n^{-1}}{\mathrm{d}u_n} \end{pmatrix}$

3 Examples

1. Let X_1, X_2 be i.i.d. U$(0, 1)$ random variables. Find the joint pdf of U_1, U_2 where $U_1 = X_1 + X_2$ and $U_2 = X_1 - X_2$

We see that since $U_1 = X_1 + X_2$ and $U_2 = X_1 - X_2$, we can re-write these equations as

$$\begin{aligned} X_1 &= \frac{U_1 + U_2}{2} \\ X_2 &= \frac{U_1 - U_2}{2} \end{aligned}$$

This means that

$$\begin{aligned} J &= det \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \\ &= -\frac{1}{4} - \frac{1}{4} \\ &= -\frac{1}{2} \end{aligned}$$

Since X_1, X_2 are i.i.d. U$(0,1)$ we see that the joint pdf of X_1, X_2 is

$$f(x_1, x_2) = \begin{cases} 1 & 0 < x_1, x_2 < 1 \\ 0 & else \end{cases}$$

That means that

$$\begin{aligned} f(u_1, u_2) &= f_{X_1, X_2}(\frac{U_1 + U_2}{2}, \frac{U_1 - U_2}{2})|J| \\ &= \begin{cases} 1|-\frac{1}{2}| & 0 < \frac{U_1+U_2}{2} < 1 \text{ and } 0 < \frac{U_1-U_2}{2} < 1 \\ 0 & else \end{cases} \\ &= \begin{cases} \frac{1}{2} & -1 < U_2 < 1 \text{ and } |U_2| < U_1 < 2 - |U_2| \\ 0 & else \end{cases} \end{aligned}$$

Side Note: Determining the bounds on U_1, U_2

The trick to setting up bounds on U_1, U_2 is to get the bound on one that is free of the other and then get the bound on the other in terms of the one that you have already solved for. So, for the first one we take the difference of the two bounds given:

$$\begin{aligned} & 0 < \frac{U_1 + U_2}{2} < 1 \\ - \quad & 0 < \frac{U_1 - U_2}{2} < 1 \\ = \quad & 0 < \frac{U_1 + U_2}{2} < 1 \end{aligned}$$

$$
\begin{aligned}
+ & \quad -1 < \frac{-U_1 + U_2}{2} < 0 \\
= & \quad -1 < U_2 < 1
\end{aligned}
$$

for the second one we solve both original inequalities in terms of U_2

$$
\begin{aligned}
0 < \frac{U_1 + U_2}{2} < 1 & \quad \Rightarrow \quad -U_2 < U_1 < 2 - U_2 \\
0 < \frac{U_1 - U_2}{2} < 1 & \quad \Rightarrow \quad U_2 < U_1 < 2 + U_2
\end{aligned}
$$

This means that $-U_2 < U_1$ and $U_2 < U_1$, so $\max(-U_2, U_2) < U_1 \Rightarrow |U_2| < U_1$. Additionally $U_1 < 2 - U_2$ and $U_1 < 2 + U_2$, so $U_1 <\min(2 - U_2, 2 + U_2) \Rightarrow U_1 < 2 - |U_2|$

2. Let $W = X_1/X_2$. Find the pdf of W.

Finding the distribution of the ratio of random variables can be difficult, but using the multivariate jacobian techniques can provide one solution.

First we set up a "dummy" variable so that we can do a one-to-one transformation. Let $W_1 = W$ and $W_2 = X_2$.

Now we re-write these equations for X_1, X_2

$$
\begin{aligned}
X_1 &= W_1 W_2 \\
X_2 &= W_2
\end{aligned}
$$

This means that

$$
\begin{aligned}
J &= det \begin{pmatrix} W_2 & W_1 \\ 0 & 1 \end{pmatrix} \\
&= w_2(1) - w_1(0) \\
&= w_2
\end{aligned}
$$

So, like before, we have

That means that

$$
f(w_1, w_2) = f_{X_1,X_2}(w_1 w_2, w_2)|J|
$$

$$= \begin{cases} 1|w_2| & 0 < w_1 w_2 < 1 \text{ and } 0 < w_2 < 1 \\ 0 & else \end{cases}$$

$$= \begin{cases} w_2 & 0 < w_1 < \frac{1}{w_2} \text{ and } 0 < w_2 < 1 \\ 0 & else \end{cases}$$

So, now we can find the pdf of $W = W_1$ by integrating the joint pdf of W_1, W_2 over the support of W_2. So,

$$\begin{aligned} f(w) &= \int_{-\infty}^{\infty} f(w_1, w_2) \mathrm{d}w_2 \\ &= \begin{cases} 0 & w_1 < 0 \\ \int_0^1 w_2 \mathrm{d}w_2 & 0 < w_1 < 1 \\ \int_0^{\frac{1}{w_1}} w_2 \mathrm{d}w_2 & 1 < w_1 \end{cases} \\ &= \begin{cases} 0 & w_1 < 0 \\ \frac{1}{2} & 0 < w_1 < 1 \\ \frac{1}{2w_1^2} & 1 < w_1 \end{cases} \end{aligned}$$

4 Exercise

Let X_1, X_2 have the following joint pdf:

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} 2(1 - x_1) & 0 < x_1 < 1 \text{ and } 0 < x_2 < 1 \\ 0 & else \end{cases}$$

Find the pdf of $U = X_1 X_2$

5 Solution

Let $U_1 = X_1$ and let $U_2 = U$
This means that

$$\begin{aligned} X_1 &= U_1 \\ X_2 &= \frac{U_2}{U_1} \\ \Rightarrow J &= det \begin{pmatrix} 1 & 0 \\ -\frac{u_2}{u_1^2} & \frac{1}{u_1} \end{pmatrix} \\ &= \frac{1}{u_1} \end{aligned}$$

This means that

$$
\begin{aligned}
f(u_1,u_2) &= f_{X_1,X_2}(u_1,\frac{u_2}{u_1})|J| \\
&= \begin{cases} 2(1-u_1)|\frac{1}{u_1}| & 0<u_1<1 \text{ and } 0<\frac{u_2}{u_1}<1 \\ 0 & else \end{cases} \\
&= \begin{cases} \frac{2}{u_1}-2 & 0<u_1<1 \text{ and } 0<u_2<u_1 \\ 0 & else \end{cases}
\end{aligned}
$$

So to get the pdf of $U=U_2$ we integrate out U_1

$$
\begin{aligned}
f(u) &= \int_{-\infty}^{\infty} f(u_1,u_2)\mathrm{d}u_1 \\
&= \begin{cases} \int_{u_2}^{1} \frac{2}{u_1}-2\mathrm{d}u_1 & 0<u_2<1 \\ 0 & else \end{cases} \\
&= \begin{cases} 2(u_2-ln(u_2)-1) & 0<u_2<1 \\ 0 & else \end{cases}
\end{aligned}
$$