Homework 7

SOLUITONS!

1. Evaluate $\int_{-\infty}^{\infty}(x+1)^2e^{-x^2}\mathrm{d}x$
   Solution:

$$
\begin{aligned}
\int_{-\infty}^{\infty}(x+1)^2e^{-x^2}\mathrm{d}x &= \int_{-\infty}^{\infty}(x^2+2x+1)e^{-x^2}\mathrm{d}x \\
&= \int_{-\infty}^{\infty}x^2e^{-x^2}\mathrm{d}x+\int_{-\infty}^{\infty}2xe^{-x^2}\mathrm{d}x+\int_{-\infty}^{\infty}1e^{-x^2}\mathrm{d}x \\
&= 2\int_{0}^{\infty}x^2e^{-x^2}\mathrm{d}x+0+2\int_{0}^{\infty}e^{-x^2}\mathrm{d}x \\
& \quad \uparrow \text{Since } 2xe^{-x^2} \text{ is an odd function} \\
& \quad \text{Integration by Substitution} \\
u=x^2 &\Rightarrow x=\sqrt{u} \\
\Rightarrow \mathrm{d}u=2x\mathrm{d}x & \quad \frac{\mathrm{d}u}{2\sqrt{u}}=\mathrm{d}x \\
\Rightarrow 2\int_{0}^{\infty}x^2e^{-x^2}\mathrm{d}x+2\int_{0}^{\infty}e^{-x^2}\mathrm{d}x &= 2\int_{0}^{\infty}ue^{-u}\frac{\mathrm{d}u}{2\sqrt{u}}+2\int_{0}^{\infty}e^{-u}\frac{\mathrm{d}u}{2\sqrt{u}} \\
&= \int_{0}^{\infty}\sqrt{u}e^{-u}\mathrm{d}u+\int_{0}^{\infty}u^{-1/2}e^{-u}\mathrm{d}u \\
&= \int_{0}^{\infty}u^{3/2-1}e^{-u}\mathrm{d}u+\int_{0}^{\infty}u^{1/2-1}e^{-u}\mathrm{d}u \\
&= \Gamma(\frac{3}{2})+\Gamma(\frac{1}{2}) \\
&= \frac{1}{2}\sqrt{\pi}+\sqrt{\pi} \\
&= \frac{3}{2}\sqrt{\pi}
\end{aligned}
$$

2. Let $X \sim N(\mu, \sigma^2)$. Derive the variance of X without using the Moment Generating Function of X
   Solution:

$$
\begin{aligned}
V[X] &= E[(X-E[X])^2] \\
&= E[(X-\mu)^2] \\
&= \int_{-\infty}^{\infty}(X-\mu)^2 f_X(x)\mathrm{d}x \\
&= \int_{-\infty}^{\infty}(X-\mu)^2\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\mathrm{d}x \\
&= \int_{-\infty}^{\infty}\frac{\sqrt{2}\sigma}{\sqrt{2}\sigma}\frac{(X-\mu)^2}{\sqrt{2}\sigma}\frac{1}{\sqrt{\pi}}e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2}\mathrm{d}x \\
&= \sqrt{2}\sigma\int_{-\infty}^{\infty}(\frac{X-\mu}{\sqrt{2}\sigma})^2\frac{1}{\sqrt{\pi}}e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2}\mathrm{d}x \\
& \quad \text{Integration by substitution} \\
u=\frac{x-\mu}{\sqrt{2}\sigma} &\Rightarrow \mathrm{d}u=\frac{\mathrm{d}x}{\sqrt{2}\sigma} \\
&\Rightarrow \sqrt{2}\sigma\mathrm{d}u=\mathrm{d}x \\
\Rightarrow u \text{ Ranges from } -\infty \text{ to } \infty & \quad \text{When } x \text{ ranges from } -\infty \text{ to } \infty \\
\Rightarrow \sqrt{2}\sigma\int_{-\infty}^{\infty}(\frac{X-\mu}{\sqrt{2}\sigma})^2\frac{1}{\sqrt{\pi}}e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2}\mathrm{d}x &= \sqrt{2}\sigma\int_{-\infty}^{\infty}(u)^2\frac{1}{\sqrt{\pi}}e^{-(u)^2}\sqrt{2}\sigma\mathrm{d}u \\
&= \frac{2\sigma^2}{\sqrt{\pi}}\int_{-\infty}^{\infty}(u)^2e^{-(u)^2}\mathrm{d}u \\
&= \frac{2\sigma^2}{\sqrt{\pi}}2\int_{0}^{\infty}(u)^2e^{-(u)^2}\mathrm{d}u \\
& \quad \uparrow \text{Since } (u)^2e^{-(u)^2} \text{ is an even function} \\
& \quad \text{Integration by substitution} \\
v=u^2 &\Rightarrow \mathrm{d}v=2u\mathrm{d}u \\
\Rightarrow \sqrt{v}=u &\Rightarrow \frac{\mathrm{d}v}{2\sqrt{v}}=\mathrm{d}u \\
\Rightarrow v \text{ Ranges from } 0 \text{ to } \infty & \quad \text{When } u \text{ ranges from } 0 \text{ to } \infty \\
\Rightarrow \frac{2\sigma^2}{\sqrt{\pi}}2\int_{0}^{\infty}(u)^2e^{-(u)^2}\mathrm{d}u &= \frac{2\sigma^2}{\sqrt{\pi}}2\int_{0}^{\infty}ve^{-v}\frac{\mathrm{d}v}{2\sqrt{v}} \\
&= \frac{2\sigma^2}{\sqrt{\pi}}\int_{0}^{\infty}\sqrt{v}e^{-v}\mathrm{d}v \\
&= \frac{2\sigma^2}{\sqrt{\pi}}\int_{0}^{\infty}v^{3/2-1}e^{-v}\mathrm{d}v \\
&= \frac{2\sigma^2}{\sqrt{\pi}}\Gamma(3/2) \\
&= \frac{2\sigma^2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2}
\end{aligned}
$$

$$= \quad \sigma^2$$

3. Let $X \sim N(\mu, \sigma^2)$. Derive the Moment Generating Function of X
Solution:

$$\begin{aligned}
M_X(t) &= E[e^{Xt}] \\
&= \int_{-\infty}^{\infty} e^{xt} f_X(x)\mathrm{d}x \\
&= \int_{-\infty}^{\infty} e^{xt} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \mathrm{d}x \\
&= \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(-2xt\sigma^2)} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x^2-2x\mu+\mu^2-2xt\sigma^2)} \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x^2-2x(\mu+t\sigma^2)+\mu^2} \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x^2-2x(\mu+t\sigma^2)+(\mu+t\sigma^2)^2-(\mu+t\sigma^2)^2+\mu^2} \mathrm{d}x \\
&\quad \text{completing the square} \uparrow \\
&= \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(-(\mu+t\sigma^2)^2+\mu^2)} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}\left(x-(\mu+t\sigma^2)\right)^2} \mathrm{d}x \\
&= e^{\frac{(\mu+t\sigma^2)^2-\mu^2}{2\sigma^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}\left(x-(\mu+t\sigma^2)\right)^2} \mathrm{d}x \\
&\quad \text{This function here } \uparrow \text{ is the pdf of } X \sim N(\mu - t\sigma^2, \sigma^2) \\
&= e^{\frac{\mu^2+2\mu t\sigma^2+(t\sigma^2)^2-\mu^2}{2\sigma^2}}(1) \\
&= e^{\frac{2\mu t\sigma^2+(t\sigma^2)^2}{2\sigma^2}} \\
&= e^{\mu t+\frac{t^2}{2}\sigma^2}
\end{aligned}$$

4. Let $X \sim N(\mu, \sigma^2)$. Derive the third Moment of X (i.e. derive $E[X^3]$).
Solution:

$$\begin{aligned}
E[X^3] &= \frac{\mathrm{d}^3}{\mathrm{d}t^3} M_X(t)|_{t=0} \\
&= \frac{\mathrm{d}^3}{\mathrm{d}t^3} e^{\mu t+\frac{t^2}{2}\sigma^2}|_{t=0} \\
&= \frac{\mathrm{d}^2}{\mathrm{d}t^2} [e^{\mu t+\frac{t^2}{2}\sigma^2}(\mu+t\sigma^2)]_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} [e^{\mu t+\frac{t^2}{2}\sigma^2}(\mu+t\sigma^2)^2 + e^{\mu t+\frac{t^2}{2}\sigma^2}(\sigma^2)]_{t=0}
\end{aligned}$$

$$
\begin{aligned}
&= [e^{\mu t+\frac{t^2}{2}\sigma^2}(\mu+t\sigma^2)^3 + e^{\mu t+\frac{t^2}{2}\sigma^2}2(\mu+t\sigma^2)(\sigma^2) + e^{\mu t+\frac{t^2}{2}\sigma^2}(\sigma^2)(\mu+t\sigma^2)]_{t=0} \\
&= [e^{\mu(0)+\frac{(0)^2}{2}\sigma^2}(\mu+(0)\sigma^2)^3 + e^{\mu(0)+\frac{(0)^2}{2}\sigma^2}2(\mu+(0)\sigma^2)(\sigma^2) + e^{\mu(0)+\frac{(0)^2}{2}\sigma^2}(\sigma^2)(\mu+(0)\sigma^2)] \\
&= [\mu^3 + 2\mu\sigma^2 + \mu\sigma^2] \\
&= \mu^3 + 3\mu\sigma^2
\end{aligned}
$$