Homework 1

SOLUTIONS!

1. Problem 2.6 from the book (p 26)
   Solution:
   $A = \{(1,2), (2,2), (3,2), (4,2), (5,2), (6,2), (1,4), (2,4), (3,4), (4,4), (5,4), (6,4), (1,6), (2,6), (3,6), (4,6), (5,6), (6,6)\}$
   $\overline{C} = \{(2,2), (2,4), (2,6), (4,2), (4,4), (4,6), (6,2), (6,4), (6,6)\}$
   $A \cap B = \{(2,2), (4,2), (6,2), (2,4), (4,4), (6,4), (2,6), (4,6), (6,6)\}$
   $A \cap \overline{B} = \{(1,2), (3,2), (5,2), (1,4), (3,4), (5,4), (1,6), (3,6), (5,6)\}$
   $\overline{A} \cup B =$ everything but $\{(1,2), (1,4), (1,6), (3,2), (3,4), (3,6), (5,2), (5,4), (5,6)\}$
   $\overline{A} \cap C = \overline{A}$

2. Let A and B both be subsets of the superset S. Prove that $(A \cap B)^C = A^C \cup B^C$
   Solution:
   To prove that these sets are equal we must show that
   a) $(A \cap B)^C \subset A^C \cup B^C$
   b) $A^C \cup B^C \subset (A \cap B)^C$

Proof. a) We want to show that $(A \cap B)^C \subset A^C \cup B^C$
Let $a \in (A \cap B)^C$.

$$\Rightarrow \quad a \notin A \cap B$$
$$\Rightarrow \quad a \notin A \text{ or } a \notin B$$
$\uparrow$ If a were an element of A and B, then a would be an element of $A \cap B$
$$\Rightarrow \quad a \in A^C \text{ or } a \in B^C$$
$$\Rightarrow \quad a \in A^C \cup B^C$$

Thus, $(A \cap B)^C \subset A^C \cup B^C$.

b) Now, we want to show that $A^C \cup B^C \subset (A \cap B)^C$

Let $a \in A^C \cup B^C$

$$\begin{aligned}
&\Rightarrow \quad a \in A^C \text{ or } a \in B^C \\
&\Rightarrow \quad a \notin A \text{ or } a \notin B \\
&\Rightarrow \quad a \notin A \cap B \\
&\Rightarrow \quad a \in (A \cap B)^C
\end{aligned}$$

Thus, $A^C \cup B^C \subset (A \cap B)^C$.

Therefore $(A \cap B)^C = A^C \cup B^C$. □

3. Let A and B be events from sample space S such that $B \subset A$. Prove that

$$P(A) = P(B) + P(A \cap B^C)$$

Solution

Proof. First, we will show that $A = B \cup (A \cap B^C)$.

$$\begin{aligned}
A &= A \cap S \leftarrow \text{Since } A \subset S \\
&= A \cap (B \cup B^C) \leftarrow \text{By the definition of set compliment} \\
&= (A \cap B) \cup (A \cap B^C) \leftarrow \text{By the distributive law} \\
&= B \cup (A \cap B^C) \leftarrow \text{Since } B \subset A
\end{aligned}$$

Now, we will show that B and $(A \cap B^C)$ are disjoint

$$\begin{aligned}
B \cap (A \cap B^C) &= B \cap (B^C \cap A) \leftarrow \text{By the law of commutativeity} \\
&= (B \cap B^C) \cap A \leftarrow \text{By the law of associativity} \\
&= (\varnothing) \cap A \leftarrow \text{By definition of set compliment} \\
&= \varnothing
\end{aligned}$$

So, now we can conclude that

$$\begin{aligned}
P(A) &= P(B \cup (A \cap B^C)) \\
&= P(B) + P(A \cap B^C) \leftarrow \text{By the } 4^{th} \text{ rule of probability}
\end{aligned}$$

□