The Normal Distribution

1 Definition & Motivation

- Known by several names including Normal, Gaussian, Bell curve/distribution (here we will call it the Normal distirbution), the Normal distribution is probably the most influential and impactful distribution in all of statistics and Science
- In addition to being a fairly good approximation of distributions found in nature/real life, the Central Limit Theorem (which is covered in the Spring semester) tells us that under certain conditions the average of several random variables is normally distriubted, regardless of what the original distribution of the original random variables is

  Definition 1. *We say that the continuous Random Variable X is normally distributed with parameters μ and σ^2 when X has the PDF*

  $$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2}$$

  or alternatively,

  $$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

  For any Real valued x, Where $-\infty < \mu < \infty$, $0 < \sigma^2 < \infty$. We denoted this $X \sim N(\mu, \sigma^2)$

2 Notes

- For our proofs of the verification, Mean, & Variance of the Normal distirbutions, we will be making use of a few mathematical properties of functions and integration
- First, we will define odd and eve functions:

  Definition 2. *A function g is an* odd *function if $-g(x) = g(-x)$ for all real values of x.*

  Definition 3. *A function g is an* even *function if $g(x) = g(-x)$ for all real values of x.*

- Functions of these forms can make integration easier, which we will highlight in the following theorems:

  Theorem 1. *Let g be an even function defined for all real values, and let a be a real constant. Then*

  $$\int_{-a}^{a} g(x)\mathrm{d}x = 2\int_{0}^{a} g(x)\mathrm{d}x$$

Proof.

$$
\begin{aligned}
\int_{-a}^{a} g(x)\mathrm{d}x &= \int_{-a}^{0} g(x)\mathrm{d}x + \int_{0}^{a} g(x)\mathrm{d}x \\
&= -\int_{0}^{-a} g(x)\mathrm{d}x + \int_{0}^{a} g(x)\mathrm{d}x \\
\text{Integration by substitution} & \\
u = -x &\Rightarrow -u = x \\
&\Rightarrow -\mathrm{d}u = \mathrm{d}x \\
\Rightarrow u \text{ Ranges from } 0 \text{ to } a & \quad \text{When } x \text{ ranges from } 0 \text{ to } -a \\
\Rightarrow -\int_{0}^{-a} g(x)\mathrm{d}x + \int_{0}^{a} g(x)\mathrm{d}x &= -\int_{0}^{a} g(-u) - \mathrm{d}u + \int_{0}^{a} g(x)\mathrm{d}x \\
&= \int_{0}^{a} g(-u)\mathrm{d}u + \int_{0}^{a} g(x)\mathrm{d}x \\
&= \int_{0}^{a} g(u)\mathrm{d}u + \int_{0}^{a} g(x)\mathrm{d}x \leftarrow \text{Since } g \text{ is an even function} \\
&= 2\int_{0}^{a} g(x)\mathrm{d}x
\end{aligned}
$$

□

Theorem 2. *Let g be an odd function defined for all real values, and let a be a real constant. Then*

$$\int_{-a}^{a} g(x)\mathrm{d}x = 0$$

Proof.

$$
\begin{aligned}
\int_{-a}^{a} g(x)\mathrm{d}x &= \int_{-a}^{0} g(x)\mathrm{d}x + \int_{0}^{a} g(x)\mathrm{d}x \\
&= -\int_{0}^{-a} g(x)\mathrm{d}x + \int_{0}^{a} g(x)\mathrm{d}x \\
\text{Integration by substitution} & \\
u = -x &\Rightarrow -u = x \\
&\Rightarrow -\mathrm{d}u = \mathrm{d}x \\
\Rightarrow u \text{ Ranges from } 0 \text{ to } a & \quad \text{When } x \text{ ranges from } 0 \text{ to } -a \\
\Rightarrow -\int_{0}^{-a} g(x)\mathrm{d}x + \int_{0}^{a} g(x)\mathrm{d}x &= -\int_{0}^{a} g(-u) - \mathrm{d}u + \int_{0}^{a} g(x)\mathrm{d}x \\
&= \int_{0}^{a} g(-u)\mathrm{d}u + \int_{0}^{a} g(x)\mathrm{d}x \\
&= -\int_{0}^{a} g(u)\mathrm{d}u + \int_{0}^{a} g(x)\mathrm{d}x \leftarrow \text{Since } g \text{ is an odd function}
\end{aligned}
$$

$$= \quad 0$$

$\square$

3 Verify, Mean, Variance, & MGF

1. Verify

   a) $f_X(x) \geq 0 \forall x \in S$

   Proof. We see that $\frac{1}{\sqrt{2\pi}\sigma} > 0$ by definition and $e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2} > 0$ since $e > 0$. Since $f_X(x)$ is the product of these two positve terms, we see that

   $$f_X(x) \geq 0 \forall x \in S = (-\infty, \infty), \text{ the support of } X$$

   $\square$

   b) $\int_{-\infty}^{\infty} f_X(x)\mathrm{d}x = 1$

   Proof. Using integration by substitution (twice) we see

$$\begin{aligned}
\int_{-\infty}^{\infty} f_X(x)\mathrm{d}x &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2} \mathrm{d}x \\
u = \frac{x-\mu}{\sqrt{2}\sigma} & \quad \mathrm{d}u = \frac{1}{\sqrt{2}\sigma}\mathrm{d}x \\
\Rightarrow u \text{ Ranges from } -\infty \text{ to } \infty & \quad \text{When } x \text{ ranges from } -\infty \text{ to } \infty \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-(u)^2} \mathrm{d}u \\
&= 2\int_{0}^{\infty} \frac{1}{\sqrt{\pi}} e^{-(u)^2} \mathrm{d}u \\
& \quad \uparrow \text{Since } \tfrac{1}{\sqrt{\pi}} e^{-(u)^2} \text{ is an even function} \\
&= \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-(u)^2} \mathrm{d}u \\
v = u^2 & \quad \mathrm{d}v = 2u\mathrm{d}u \\
&\Rightarrow \frac{\mathrm{d}v}{2\sqrt{v}} = \mathrm{d}u \\
\Rightarrow v \text{ Ranges from } 0 \text{ to } \infty & \quad \text{When } u \text{ ranges from } 0 \text{ to } \infty \\
\Rightarrow \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-(u)^2} \mathrm{d}u &= \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{v}} e^{-v} \frac{\mathrm{d}v}{2} \\
&= \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} v^{1/2-1} e^{-v} \mathrm{d}v
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{\pi}}\Gamma(\frac{1}{2}) \\
&= \frac{1}{\sqrt{\pi}}\sqrt{\pi} \\
&= 1
\end{aligned}
$$

$\square$

2. Mean

$$E[X] = \mu$$

Proof. Using integration by substitution (twice) we see

$$
\begin{aligned}
\int_{-\infty}^{\infty} x f_X(x)\mathrm{d}x &= \int_{-\infty}^{\infty} x\frac{1}{\sqrt{2\pi}\sigma}e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2}\mathrm{d}x \\
&= \int_{-\infty}^{\infty} (x-\mu+\mu)\frac{1}{\sqrt{2\pi}\sigma}e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2}\mathrm{d}x \\
&= \int_{-\infty}^{\infty} (x-\mu)\frac{1}{\sqrt{2\pi}\sigma}e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2}\mathrm{d}x + \int_{-\infty}^{\infty} \mu\frac{1}{\sqrt{2\pi}\sigma}e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2}\mathrm{d}x \\
&= \int_{-\infty}^{\infty} (x-\mu)\frac{1}{\sqrt{2\pi}\sigma}e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2}\mathrm{d}x + E[\mu] \\
&= \int_{-\infty}^{\infty} \frac{x-\mu}{\sqrt{2}\sigma}\frac{1}{\sqrt{\pi}}e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2}\mathrm{d}x + \mu \\
u = \frac{x-\mu}{\sqrt{2}\sigma} & \qquad \mathrm{d}u = \frac{1}{\sqrt{2}\sigma}\mathrm{d}x \\
&\Rightarrow \mathrm{d}x = \sigma\sqrt{2}\mathrm{d}u \\
\Rightarrow u \text{ Ranges from } -\infty \text{ to } \infty & \qquad \text{When } x \text{ ranges from } -\infty \text{ to } \infty \\
&= \int_{-\infty}^{\infty} u\frac{1}{\sqrt{\pi}}e^{-(u)^2}\sqrt{2}\sigma\mathrm{d}u + \mu \\
&= \sqrt{2}\sigma\int_{-\infty}^{\infty} u\frac{1}{\sqrt{\pi}}e^{-(u)^2}\mathrm{d}u + \mu
\end{aligned}
$$

Note, $u\frac{1}{\sqrt{\pi}}e^{-(u)^2}$ is an odd function. (i.e. if we call $u\frac{1}{\sqrt{\pi}}e^{-(u)^2} = g(u)$, then $g(-u) = -g(u)$). So, we see that

$$
\begin{aligned}
\sqrt{2}\sigma\int_{-\infty}^{\infty} u\frac{1}{\sqrt{\pi}}e^{-(u)^2}\mathrm{d}x &= \sqrt{2}\sigma\int_{-\infty}^{\infty} g(u)\mathrm{d}u \\
&= \sqrt{2}\sigma\int_{-\infty}^{0} g(u)\mathrm{d}u + \sqrt{2}\sigma\int_{0}^{\infty} g(u)\mathrm{d}u
\end{aligned}
$$

$$
\begin{aligned}
&= \sqrt{2}\sigma \int_0^\infty g(-u)\mathrm{d}u + \sqrt{2}\sigma \int_0^\infty g(u)\mathrm{d}u \\
&= -\sqrt{2}\sigma \int_0^\infty g(u)\mathrm{d}u + \sqrt{2}\sigma \int_0^\infty g(u)\mathrm{d}u \\
&= 0
\end{aligned}
$$

So, since $\sigma\sqrt{2}\int_{-\infty}^{\infty} u\frac{1}{\sqrt{\pi}}e^{-(u)^2}\mathrm{d}x = 0$, we see that

$$
\begin{aligned}
\int_{-\infty}^{\infty} x f_X(x)\mathrm{d}x &= \sigma\sqrt{2}\int_{-\infty}^{\infty} u\frac{1}{\sqrt{\pi}}e^{-(u)^2}\mathrm{d}u + \mu \\
&= 0 + \mu \\
&= \mu
\end{aligned}
$$

□

3. Variance

$$V[X] = \sigma^2$$

Proof. Proof Left as HW □

4. MGF

$$M_X(t) = e^{t\mu + \frac{t^2}{2}\sigma^2}$$

Proof. Proof Left as HW □