1 Calculating Probabilities

- To determine the probability of an event we must add up the probabilities of the individual sample points which are elements of that event
- Sometimes figuring out the probability of a sample point is not so clear; we need a systematic way to determine the probability of a sample point
- Example: We have a bag with 3 red marbles, 2 blue marbles and 1 green marble. Let A be the event that we don't draw a red marble.

$$\begin{aligned}
\text{Sample Space } S &= \{R, B, G\}(\text{Red, Blue, and Green}) \\
A &= \{B, G\} \\
\Rightarrow P(A) &= P(\{B\}) + P(\{G\})
\end{aligned}$$

- So, we need to find $P(\{B\})$ and $P(\{G\})$.
- In cases like this, we can find the probability of a sample point by counting
- In general, P(sample point E) $= \frac{\text{\# ways that sample point } E \text{ can happen}}{\text{\# of possible outcomes of the experiment}}$
- So, this means that $P(\{B\}) = \frac{\text{\# ways that we can draw a blue marble}}{\text{\# of possible outcomes of the experiment}}$ and $P(\{G\}) = \frac{\text{\# ways that we can draw a green marble}}{\text{\# of possible outcomes of the experiment}}$
- To find the # of possible outcomes of the experiment, we will need to find the # of ways we can draw a red marble, the # of ways we can draw a blue marble, and the # of ways that we can draw a green marble (basically we need to find the number of ways each sample point can happen)
- Here is where we will need the combination and permutation techniques

2 Permutation and Combination

- In counting the number of ways a sample point can happen, there are two main components
  1. Counting the ways we can *select* things from a larger pool of things that can fulfill the sample point description
  2. Counting the number of ways that we can *order* that selection to properly meet the description of the sample point
- To calculate the number of ways we can *select* k things from a larger group of n equivalent (with respect to fulfilling the sample point description) things is called combination, and is denoted

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Where $m! = m \times (m-1) \times (m-2) \times ... \times 2 \times 1$, where m is any positive whole number. Note: $0! = 1$

- When the larger group of possible things is not equivalent, each case must be accounted for separately; combination can not be directly applied.
- To calculate the ways that we can order these k things after we have selected them simply is $k!$ (Assuming that all orders are valid for the sample point in question). This technique is called *Permutation*.
- To calculate the total number of ways a sample point can happen we will calculate the number of ways we can select the appropriate objects from a larger pool of objects, and then multiply that number by the number of ways we can order those k objects (In a way that meets the description of the sample point)
- Sometimes, the order of our selection does not matter. In these cases we only need to multiply our result by 1, but if the order does not matter and you account for it anyway, your resulting solutions will still be correct. Therefore, it is usually best to account for order.

So, now lets return to our first example.

- The number of ways we can draw a blue marble will be the number of ways that we can select 1 blue marble from the possible 2 blue marbles in the bag times 1, since the number of ways that we can order that one marble doesn't matter
  - This is $\binom{2}{1} = \frac{2!}{1!(2-1)!} = 2$
- The number of ways we can draw a red marble will be the number of ways that we can select 1 red marble from the possible 3 red marbles in the bag times 1, since the number of ways that we can order that one marble doesn't matter
  - This is $\binom{3}{1} = \frac{3!}{1!(3-1)!} = \frac{3!}{2!} = 3$
- The number of ways we can draw a green marble will be the number of ways that we can select 1 green marble from the possible 1 green marble times 1, since the number of ways that we can order that one marble doesn't matter
  - This is $\binom{1}{1} = \frac{1!}{1!(0)!} = 1$
- This means that the # of possible outcomes of the experiment $= 3 + 2 + 1 = 6$
- Therefore $P(\{B\}) = 2/6 = 1/3$ and $P(\{G\}) = 1/6$ and $P(A) = 2/6 + 1/6 = 3/6 = 1/2$

Now, lets consider a slightly more complicated problem. Imagine we have the same bag, but now we will draw two marbles from the bag; one at a time and without replacement. Let B be the event that we draw a red marble and then a green marble. What is the probability of B?

- First, we construct a new sample space to reflect the new experiment. We can use ordered tuple (in this case, ordered pair) notation to indicate the order of things happening. For example, (R, G) would be hte sample point where a red marble was drawn first, followed by a green marble. Then the resulting sample space would be:

$$S = \left\{ \begin{array}{lll} (R, R), & (R, B), & (R, G), \\ (B, R), & (B, B), & (B, G), \\ (G, R), & (G, B) & \end{array} \right\}$$

- Note, (G, G) is not a sample point because the green marble cannot be drawn twice.

- We can see that the even that we draw a red marble and then a green marble is a simple event because drawing a red marble and then a green marble is a distinct possible outcome of the experiment, so we only need to count the number of ways we can draw a red marble and then a green marble and the number of ways that we can draw two marbles in general

- Sometimes in cases such as this, a sample point can be broken down into parts for counting purposes

- Let consider how we can count the number of ways this sample point can happen
  - The sample point consists of two parts:
    1. Drawing a red marble
    2. Then drawing a green marble
  - That is, we can decompose the "number of ways to select" portion of our calculations into two smaller counting problems. We calculate the number of ways a red marble can be selected and the number of ways a green marble can be selected and then multiply the results to give us the number of was we can select a red and green marble.
  - Here, what we are seeing is how counting problems can have nested components to them; it is important to decompose problems carefully when counting.
  - We can calculate the total number of ways that we can select a red marble, and a green marble by finding out how many ways we can select a red marble and how many ways we can select a green marble separately, and then multiply those number together
  - Since there are three possible ways to select the red marble and there is only one way to select the green marble from the bag, we multiply these numbers to get 3 total ways in which we can select a red and a green marble from the bag.
  - Now we need to figure out how many ways we can order these red and green marbles. While we can in fact order the two marbles two different ways, we

only want to count one of them because the sample point is described as draw a red marble and *then* a green marble

- Therefore there are three ways we can draw a red marble and then a green marble

- Next we calculate the number of ways that we can draw two marbles in general
  - Since there are 6 marbles in the bag, there are $\binom{6}{2} = 15$ ways to select two marbles from the bag
  - We see that there are $2! = 2$ ways to order these two marbles
  - Thus there are 30 ways to select two marbles from the bag
- Therefore $P(B) = 3/30 = 1/10$

Now, lets consider Event C, the event that we draw a red marble and a green marble (either red first or green first).

- We can look at this problem in two different ways
  1. We can consider event C as an event containing two sample points, one of which we have already calculated. We need only find the probability of the second one and then add the two probabilities together
  2. We can try to use the techniques described to solve for $P(C)$ directly.

  <u>Approach 1</u>

- We know that the probability of draw a red marble and then a green marble is 1/10, So we need only find the probability of drawing a green marble and then a red marble
- First we need to calculate the number of ways we can select a red marble and a green marble.
  - Since there are three possible ways to select the red marble and there is only one way to select the green marble from the bag, we multiply these numbers to get 3 total ways in which we can select a red and a green marble from the bag.
  - Now we need to figure out how many ways we can order these red and green marbles. We only want to count one of them because the sample point we are calculating for is described as draw a green marble and then a red marble
  - Therefore there are three ways we can draw a red marble and then a green marble
- We already know that the number of ways to draw two marbles from the bag is 30.
- So, the probability of drawing a green marble followed by a red marble is $3/30 = 1/10$

- Thus $P(C) = 1/10 + 1/10 = 1/5$

  Approach 2

- First we ask the number of ways that we can draw a red and green marble
  - we already know this to be 3
- Now we calculate the number of ways that we can order these two marbles to meet the *event* description. Since we can order these two marbles 2 different ways, both of which meet the event description, there are two different valid orderings
- This means there are 6 different ways to draw a red marble and a green marble
- We already know that there are 30 different ways to draw two marbles in general, so that means that $P(C) = 6/30 = 1/5$, which matches our answer from above

3 Exercises

1. Suppose that you roll two six sided dice
   a) Calculate the probability of the two dice summing to 2 (1/36)
   b) Calculate the probability of rolling two even numbers (9/36 = 1/4)
   c) Calculate the probability of the two dice summing to 3 (2/36)
   d) Calculate the probability of the two dice summing to 4 (3/36)
2. Consider a standard 52 card deck.
   a) Calculate the probability of drawing a red card (1/2)
   b) Calculate the probability of drawing a king (4/52 = 1/13)
   c) Calculate the probability of drawing a king of hearts and then the queen of hearts (1/(52*51) =1/2652)
   d) Calculate the probability of drawing a king of hearts and a queen of hearts (2/(51*52) = 1/(51*26) = 1/1326)
   e) Pretend that you are playing blackjack. Calculate the probability that your initial two cards sum to 4. Assume Aces are counted as 1. (((4*3)*1 + (4*4)*2)/(52*51) = 44 / 2652 = 11/663) ←2 2's, a 3 and an ace, or an ace and then a 3
3. Consider a bag of marbles with 3 Red marbles, 3 Green marbles, and 3 Yellow marbles.
   a) Calculate the probability of drawing the 2 yellow marbles when we draw two marbles from the bag.

$$\frac{\binom{3}{2}(2)}{\binom{9}{2}(2)} = \frac{(3)(2)}{(9)(8)(2)/2}$$

$$= \frac{1}{12}$$

b) Calculate the probability of drawing a Red and a green marble when we draw two marbles from the bag.

$$\begin{aligned} \frac{\binom{3}{1}\binom{3}{1}(2)}{\binom{9}{2}(2)} &= \frac{(3)(3)(2)}{(9)(8)(2)/2} \\ &= \frac{1}{4} \end{aligned}$$

c) Calculate the probability of drawing marbles of two different colors when we draw two marbles from the bag.

$$\begin{aligned} \frac{(\binom{3}{2}\binom{3}{1}\binom{3}{1})(2)}{\binom{9}{2}} &= \frac{(3)(3)(3)(2)}{(9)(8)(2)/2} \\ &= \frac{3}{4} \end{aligned}$$

d) Suppose that we added a red marble to the bag and removed a yellow marble from the bag. Calculate the probability of drawing marbles of two different colors when we draw two marble from the bag.

$$\begin{aligned} \frac{(\binom{4}{1}\binom{3}{1} + \binom{4}{1}\binom{2}{1} + \binom{3}{1}\binom{2}{1})(2)}{\binom{9}{2}} &= \frac{[(4)(3) + (4)(2) + (3)(2)](2)}{(9)(8)(2)/2} \\ &= \frac{[12 + 8 + 6](2)}{72} \\ &= \frac{52}{72} \\ &= \frac{13}{18} \end{aligned}$$