Homework 12

SOLUTIONS!

1. Problem 10.23 from the book (p 505)

   Solution:

   a.-b. Let μ_1 and μ_2 denote the mean distances. Since there is no prior knowledge, we will perform the test H_0: $\mu_1 - \mu_2 = 0$ vs. H_a: $\mu_1 - \mu_2 \neq 0$, which is a two–tailed test.

   c. The computed test statistic is $z = -.954$, which does not lead to a rejection with $\alpha = .10$: there is not enough evidence to conclude the mean distances are different.

2. Problem 10.25 from the book (p 505)

   Solution:

   Let p = proportion of adults who always vote in presidential elections. Then, H_0: $p = .67$, H_a: $p \neq .67$ and the large sample test statistic for a proportion is $|z| = 1.105$. With $z_{.005} = 2.576$, the null hypothesis cannot be rejected: there is not enough evidence to conclude the reported percentage is false.

3. Problem 10.44 from the book (p 510)

   Solution:

   We require $\alpha = P(\overline{Y}_1 - \overline{Y}_2 > c \mid \mu_1 - \mu_2 = 0) = P\left(Z > \frac{c-0}{\sqrt{(\sigma_1^2+\sigma_2^2)/n}}\right)$, so that $z_\alpha = \frac{c\sqrt{n}}{\sqrt{\sigma_1^2+\sigma_2^2}}$. Also, $\beta = P(\overline{Y}_1 - \overline{Y}_2 \le c \mid \mu_1 - \mu_2 = 3) = P\left(Z \le \frac{(c-3)\sqrt{n}}{\sqrt{\sigma_1^2+\sigma_2^2}}\right)$, so that $-z_\beta = \frac{(c-3)\sqrt{n}}{\sqrt{\sigma_1^2+\sigma_2^2}}$. By eliminating c in these two expressions, we have $z_\alpha\sqrt{\frac{\sigma_1^2+\sigma_2^2}{n}} = 3 - z_\beta\sqrt{\frac{\sigma_1^2+\sigma_2^2}{n}}$. Solving for n, we have

   $$n = \frac{2(1.645)^2[(4.34)^2+(4.56)^2]}{3^2} = 47.66\,.$$

   A sample size of 48 will provide the required levels of α and β.

4. Problem 10.46 from the book (p 512)

   Solution:

   The rejection region is $\frac{\hat{\theta}-\theta_0}{\hat{\sigma}_\theta} > z_\alpha$, which is equivalent to $\theta_0 < \hat{\theta} - z_\alpha \hat{\sigma}_{\hat{\theta}}$. The left–hand side is the $100(1-\alpha)\%$ lower confidence bound for θ.

5. Problem 10.53 from the book (p 517)

   Solution:

   a. The hypothesis of interest is H_0: $\mu_1 = 3.8$, H_a: $\mu_1 < 3.8$, where μ_1 represents the mean drop in FVC for men on the physical fitness program. With $z = -.996$, we have p–value $= P(Z < -1) = .1587$.

   b. With $\alpha = .05$, H_0 cannot be rejected.

   c. Similarly, we have H_0: $\mu_2 = 3.1$, H_a: $\mu_2 < 3.1$. The computed test statistic is $z = -1.826$ so that the p–value is $P(Z < -1.83) = .0336$.

   d. Since $\alpha = .05$ is greater than the p–value, we can reject the null hypothesis and conclude that the mean drop in FVC for women is less than 3.1.

6. Problem 10.57 from the book (p 517)

   Solution:

   Let p = proportion who renew. Then, the hypotheses are H_0: $p = .60$, H_a: $p \neq .60$. The sample proportion is $\hat{p} = 108/200 = .54$, and so the computed test statistic is $z = -1.732$. The p–value is given by $2P(Z < -1.732) = .0836$.

7. Let $X \sim Exp(\delta)$. Suppose that you want to test $H_0 : \delta = 1$ Vs. $H_a : \delta \neq 1$, and you plan to use X as your testing statistic. Let the significance of your test be fixed at α

   a) Construct an appropriate rejection region for this testing situation (*Hint:* we should want to reject if the test statistics is *either* too big *or too small*, and the probability of a Type I error on either side should be the same)
   Solution:
   $H_a : \delta \neq 1$, so our rejection region is of the form $RR = \{x : x \geq k_1 \text{ or } x \leq k_2\}$ where $P(\text{type I error}) = \alpha$ and $P(X \geq k_1 | H_0 \text{ is true}) = P(X \leq k_2 | H_0 \text{ is true})$. This implies that we want to select k_1 and k_2 such that $P(X \geq k_1 | H_0 \text{ is true}) = P(X \leq k_2 | H_0 \text{ is true}) = \alpha/2$.

$$
\begin{aligned}
\Rightarrow \alpha/2 &= P(X \leq k_2 | H_0 \text{ is true}) \\
&= P(X \leq k_2 | \delta = 1) \\
&= \int_0^{k_2} e^{-x} \mathrm{d}x \\
&= 1 - e^{-k_2} \\
\Rightarrow k_2 &= -\ln(1 - \alpha/2)
\end{aligned}
$$

   Or equivalently, k_2 is the $\alpha/2$ percentile of $Exp(1)$ And,

$$
\Rightarrow \alpha/2 = P(X \geq k_1 | H_0 \text{ is true})
$$

$$
\begin{aligned}
&= P(X \leq k_1|\delta = 1) \\
&= \int_{k_1}^{\infty} e^{-x}\mathrm{d}x \\
&= e^{-k_1} \\
\Rightarrow k_1 &= -\ln(\alpha/2)
\end{aligned}
$$

Or equivalently, k_1 is the $1-\alpha/2$ percentile of $Exp(1)$.

So, our rejection region can be characterized as either $RR = \{x : x \geq -ln(\alpha/2)$ or $x \leq -ln(1-\alpha/2)\}$ or as $RR = \{x : x \geq e_{\alpha/2}$ or $x \leq e_{1-\alpha/2}\}$ where $e_{\alpha/2}$ and $e_{1-\alpha/2}$ are the $1-\alpha/2$ and $\alpha/2$ quantiles of $Exp(1)$, respectively.

b) Suppose that the alternative hypothesis was $H_a : \delta > 1$ and you observed $X = 3$. Calculate the p-value of this observed statistic.
Solution:
The p-value is the probability that we would observe a test statistic as extreme(or more extreme) as what we actually obsereved. Here our alternative hypothesis is $H_a : \delta > 1$ so our p-value will be

$$
\begin{aligned}
P(X > 3|H_0 \text{ is true}) &= P(X > 3|\delta = 1) \\
&= \int_{3}^{\infty} e^{-x}\mathrm{d}x \\
&= 0 + e^{-3} \\
&= 1/e^3
\end{aligned}
$$

c) Based on your p-value above, would you reject or fail to reject the null hypothesis $H_0 : \delta = 1$ Vs. $H_a : \delta > 1$ at the significance level of $\alpha = .1$? (you reject if the p-value is less than the significance)
Solution:
$1/e^3 = .04978707 < .1$. So, since our p-value is less than our significance level, we would reject the null hypothesis in favor of the alternative hypothesis.