# Homework 11

**SOLUTIONS!**

1. Let $X \sim \Gamma(\alpha, \beta)$ where $\beta$ is known. Find the most powerful test with a significance of $a$ for testing $H_0 : \alpha = 1$ vs $H_a : \alpha = 1/2$.

   *Solution:*

   By the Neyman-Pearson Lemma, we know that the most powerful test rejects when

$$
\begin{aligned}
& \frac{L(1)}{L(1/2)} < k \\
\iff & \frac{(\Gamma(1)\beta^1)^{-1}exp(-x/\beta)}{\beta)^{-1}x^{-1/2}exp(-x/\beta)} < k \\
\iff & \frac{(\Gamma(1)\beta^1)^{-1}}{(\Gamma(1/2)\beta^{1/2})^{-1}x^{-1/2}} < k \leftarrow \text{algebraic reduction of fraction} \\
\iff & \frac{1}{x^{-1/2}} < k \leftarrow \text{divide both sides by constant involving } \alpha \text{ and } \beta \\
\iff & x^{1/2} < k \\
\iff & x < k
\end{aligned}
$$

   So, the most powerful test will reject the null hypothesis when $X < k$ for an appropriately selected $k$. Since the significance is $a$, we know that

$$
\begin{aligned}
a &= P(\text{Type I Error}) \\
&= P(X < k | \delta = 1) \\
&= \int_0^k e^{-x} \mathrm{d}x \\
&= -e^{-x}|_0^k \\
&= 1 - e^{-k} \\
\Rightarrow k &= -\ln(1 - a)
\end{aligned}
$$

   Therefore, the most powerful test rejects the null hypothesis when $X < -\ln(1 - a)$

2. Let $X \sim Exp(\delta)$

   a) Show that the most powerful test for testing $H_0 : \delta = 1$ Vs. $H_a : \delta = \delta_a (> 1)$ at the $\alpha$ significance level rejects the null hypothesis in favor of the alternative when $X > -\ln(\alpha)$.

   *Solution:*
   By the Neyman-Pearson Lemma we know that the most powerful test will be of the form

$$\begin{aligned}
\frac{L(1)}{L(\delta_a)} &< k \\
\rightarrow \frac{e^{-x}}{\frac{1}{\delta_a} e^{-x/\delta_a}} &< k \\
\rightarrow \delta_a e^{x(1/\delta_a - 1)} &< k \\
\rightarrow e^{x(1/\delta_a - 1)} &< k \\
\rightarrow x(1/\delta_a - 1) &< k \\
\rightarrow x &> k \leftarrow \text{Since } \delta_a > 1 \Rightarrow 1/\delta_a - 1 < 0
\end{aligned}$$

   So, the most powerful test will reject the null hypothesis when $X > k$ for an appropriately selected $k$. Since the significance is $\alpha$, we know that

$$\begin{aligned}
\alpha &= P(\text{Type I Error}) \\
&= P(X > k | \delta = 1) \\
&= \int_k^\infty e^{-x} \mathrm{d}x \\
&= -e^{-x}|_k^\infty \\
&= e^{-k} \\
\Rightarrow k &= -\ln(\alpha)
\end{aligned}$$

   Therefore, the most powerful test rejects the null hypothesis when $X > -\ln(\alpha)$

   b) Show whether or not the most powerful test from part a) rejects the null hypothesis when $X = 1$ and $\alpha = .5$?

   *Solution:*
   We know that the most powerful test rejects when $X > -\ln(\alpha) = -\ln(.5) \approx 0.693147181$. So, when $X = 1$, we have that $X > -\ln(.5)$, therfore we see that the test would reject the null hypothesis.

3. Let $X_1, ..., X_n$ be i.i.d. $\Gamma(a, b)$ where $a$ is known. Suppose you would like to test $H_0 : b = 1$ Vs. $H_a : b = 3$ at the significance level $\alpha$. Derive the most powerful $\alpha$ level test for this testing situation.

*Solution:*
By the Neyman-Pearson Lemma, we know that the most powerful test will be of the form

$$\begin{aligned}
\frac{L(1)}{L(3)} &< k \\
\frac{\prod_{i=1}^{n}(\Gamma(a)1^a)^{-1}x_i^{a-1}e^{-x_i/1}}{\prod_{i=1}^{n}(\Gamma(a)3^a)^{-1}x_i^{a-1}e^{-x_i/3}} &< k \\
3^a e^{\sum_{i=1}^{n} x_i(1/3-1)} &< k \\
\sum_{i=1}^{n} x_i(1/3-1) &< k \leftarrow \text{Divide by } 3^a \text{ and take ln} \\
\sum_{i=1}^{n} x_i &> k \leftarrow \text{Divide by } 1/3-1\text{, which reverses the inequality}
\end{aligned}$$

We know that nunder the null hypothesis, $\sum_{i=1}^{n} x_i \sim \Gamma(na, 1)$, so setting the significance to be $\alpha$ we can solve for $k$:

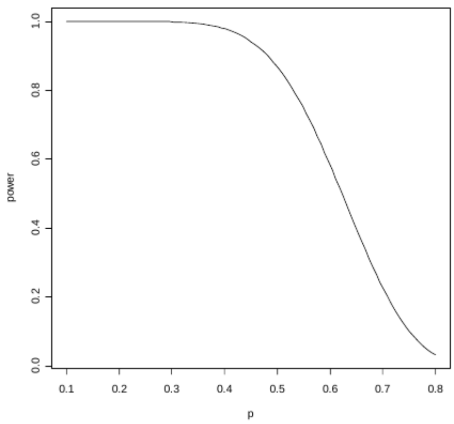
$$\begin{aligned}
P(\text{Type I error}) &= \alpha \\
\Rightarrow P(\sum_{i=1}^{n} x_i > k | b = 1) &= \alpha
\end{aligned}$$

We can see that $k$ will be a function of $a$, so when we solve for $k$, we find that $k$ is $g_\alpha$, the $1-\alpha$ percentile of $\Gamma(na, 1)$. So, the most powerful test rejects when $\sum_{i=1}^{n} X_i > g_\alpha$.

4. Problem 10.88 from the book (p 546)

*Solution:*
Refer to Ex. 10.2. Table 1 in Appendix III is used to find the binomial probabilities.

**a.** power(.4) = $P(Y \leq 12 \mid p = .4) = .979$. **b.** power(.5) = $P(Y \leq 12 \mid p = .5) = .86$
**c.** power(.6) = $P(Y \leq 12 \mid p = .6) = .584$. **d.** power(.7) = $P(Y \leq 12 \mid p = .7) = .228$




**e.** The power function is above.

5. Problem 10.89 from the book (p 546)

*Solution:*

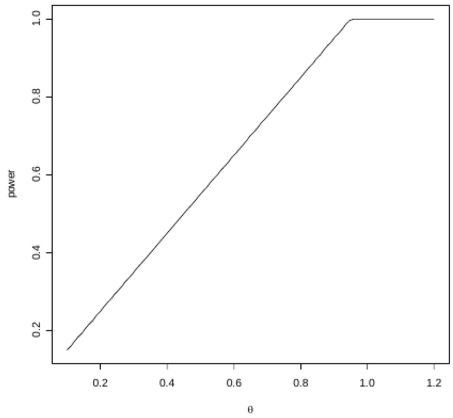
Refer to Ex. 10.5: $Y_1 \sim \text{Unif}(\theta, \theta + 1)$.

**a.** $\theta = .1$, so $Y_1 \sim \text{Unif}(.1, 1.1)$ and power(.1) $= P(Y_1 > .95) = \int_{.95}^{1.1} dy = .15$

**b.** $\theta = .4$: power(.4) $= P(Y > .95) = .45$

**c.** $\theta = .7$: power(.7) $= P(Y > .95) = .75$

**d.** $\theta = 1$: power(1) $= P(Y > .95) = 1$




**e.** The power function is above.

6. Problem 10.93 from the book (p 547)

*Solution:*

Using the sample size formula from the end of Section 10.4, we have $n = \frac{(1.96+1.96)^2(25)}{(10-5)^2} =$ 15.3664, so 16 observations should be taken.

7. Problem 10.94 from the book (p 547)

*Solution:*

The most powerful test for $H_0$: $\sigma^2 = \sigma_0^2$ vs. $H_a$: $\sigma^2 = \sigma_1^2$, $\sigma_1^2 > \sigma_0^2$, is based on the likelihood ratio:

$$\frac{L(\sigma_0^2)}{L(\sigma_1^2)} = \left(\frac{\sigma_1}{\sigma_0}\right)^n \exp\left[-\left(\frac{\sigma_1^2 - \sigma_0^2}{2\sigma_0^2\sigma_1^2}\sum_{i=1}^n (y_i - \mu)^2\right)\right] < k.$$

This simplifies to

$$T = \sum_{i=1}^n (y_i - \mu)^2 > \left[n \ln\left(\frac{\sigma_1}{\sigma_0}\right) - \ln k\right]\frac{2\sigma_0^2\sigma_1^2}{\sigma_1^2 - \sigma_0^2} = c,$$

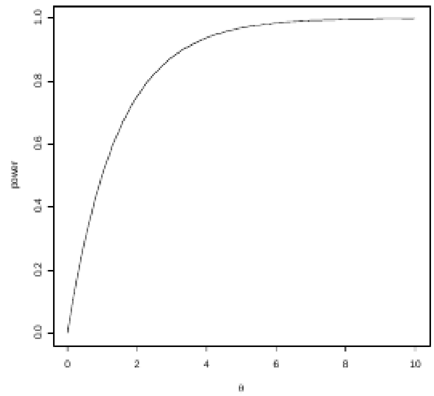
which is to say we should reject if the statistic $T$ is large. To find a rejection region of size $\alpha$, note that

$\frac{T}{\sigma_0^2} = \frac{\sum_{i=1}^n (Y_i - \mu)^2}{\sigma_0^2}$ has a chi–square distribution with $n$ degrees of freedom. Thus, the most powerful test is equivalent to the chi–square test, and this test is UMP since the RR is the same for any $\sigma_1^2 > \sigma_0^2$.

8. Problem 10.96 from the book (p 547)

*Solution:*

**a.** The power function is given by power($\theta$) = $\int_{.5}^{1} \theta y^{\theta-1} dy = 1 - .5^{\theta}$. The power function is graphed below.




**b.** To test $H_0$: $\theta = 1$ vs. $H_a$: $\theta = \theta_a$, $1 < \theta_a$, the likelihood ratio is

$$\frac{L(1)}{L(\theta_a)} = \frac{1}{\theta_a y^{\theta_a - 1}} < k.$$

This simplifies to

$$y > \left(\frac{1}{\theta_a k}\right)^{\frac{1}{\theta_a - 1}} = c,$$

where $c$ is chosen so that the test is of size $\alpha$. This is given by

$$P(Y \geq c \mid \theta = 1) = \int_c^1 dy = 1 - c = \alpha,$$

so that $c = 1 - \alpha$. Since the RR does not depend on a specific $\theta_a > 1$, it is UMP.

9. Problem 10.98 from the book (p 548)

   *Solution:*

The density function that for the Weibull with shape parameter $m$ and scale parameter $\theta$.

**a.** The best test for testing $H_0$: $\theta = \theta_0$ vs. $H_a$: $\theta = \theta_a$, where $\theta_0 < \theta_a$, is

$$\frac{L(\theta_0)}{L(\theta_a)} = \left(\frac{\theta_a}{\theta_0}\right)^n \exp\left[-\left(\frac{1}{\theta_0} - \frac{1}{\theta_a}\right)\sum_{i=1}^n y_i^m\right] < k,$$

This simplifies to

$$\sum_{i=1}^n y_i^m > -\left[\ln k + n\ln\left(\frac{\theta_0}{\theta_a}\right)\right] \times \left[\frac{1}{\theta_0} - \frac{1}{\theta_a}\right]^{-1} = c.$$

So, the RR has the form $\left\{T = \sum_{i=1}^m Y_i^m > c\right\}$, where c is chosen so the RR is of size $\alpha$. To do so, note that the distribution of $Y^m$ is exponential so that under $H_0$,

$$\frac{2T}{\theta_0} = \frac{2\sum_{i=1}^n Y_i^m}{\theta_0} > \frac{2c}{\theta_0}$$

is chi–square with $2n$ degrees of freedom. So, the critical value can be selected from the chi–square distribution and this does not depend on the specific $\theta_a > \theta_0$, so the test is UMP.

**b.** When $H_0$ is true, $T/50$ is chi–square with $2n$ degrees of freedom. Thus, $\chi^2_{.05}$ can be selected from this distribution so that the RR is $\{T/50 > \chi^2_{.05}\}$ and the test is of size $\alpha = .05$. If $H_a$ is true, $T/200$ is chi–square with $2n$ degrees of freedom. Thus, we require

$$\beta = P(T/50 \le \chi^2_{.05} \mid \theta = 400) = P(T/200 \le \tfrac{1}{4}\chi^2_{.05} \mid \theta = 400) = P(\chi^2 \le \tfrac{1}{4}\chi^2_{.05}) = .05\,.$$

Thus, we have that $\frac{1}{4}\chi^2_{.05} = \chi^2_{.95}$. From Table 6 in Appendix III, it is found that the degrees of freedom necessary for this equality is $12 = 2n$, so $n = 6$.

Challenge Question:

Let $X \sim Exp(\delta)$. Suppose that you want to test $H_0 : \delta = 1$ Vs. $H_a : \delta \neq 1$, and you plan to use $X$ as your testing statistic. Let the corresponding rejection region be $RR = \{x : x \geq -ln(\alpha/2) \text{ or } x \leq -ln(1 - \alpha/2)\}$. Calculate the probability of a type II error for this rejection region when the true value of $\delta$ is $\delta = 2$