1 Neyman-Pearson Lemma

1.1 Power of a test

Definition:
Let $X_1, ..., X_n$ be an observed set of data from a distribution that depends on one unkown parameter θ, and suppose we have some sort of hypothesis test with a test statistic U and corresponding rejection region RR. Then the power for a given value θ' is

$$Q(\theta') = P(U \in RR | \theta = \theta')$$

Note: Book uses $P(\theta')$

We have already seen the power function in the form of the significance level. i.e.

$$\begin{aligned} P(\text{Type I Error}) &= P(U \in RR | \theta = \theta_0) \\ &= Q(\theta_0) \end{aligned}$$

Similarly we can relate the probability of a type II error in terms of power

$$\begin{aligned} P(\text{Type II Error when } \theta = \theta_a) &= P(U \notin RR | \theta = \theta_a) \\ &= 1 - P(U \in RR | \theta = \theta_a) \\ &= 1 - Q(\theta_a) \end{aligned}$$

$$\Rightarrow Q(\theta_a) = 1 - P(\text{Type II Error when } \theta = \theta_a)$$

- So, when we construct a test, we fix our significance α, but how do we construct a test to guarantee that we have the best test for that significance level?
- We can answer this question in terms of the power function
- Ideally, the power of our test for any point that is in the range of possible alternative values would be 1, but this is not practically possible
- So, we characterize a test as best, or *Most Powerful* if for every possible alternative value of the parameter, the power function is larger than the power function of any other test at that same alternative value

1.2 The Lemma

Lemma:
Suppose we wish to test $H_0 : \theta = \theta_0$ vs. $H_a : \theta = \theta_a$ (so a simple vs. simple testing situation), then the most powerful test, for a given α, rejects the null in favor of the alternative hypothesis when

$$\frac{L(\theta_0)}{L(\theta_a)} < k$$

Where $L(\theta)$ is the likelihood function of the observed values from the random sample $X_1, ..., X_n$ and k is specified to give the desired significance α.

- Note that this lemma specifies the general form of the test, but not the specific testing statistic and corresponding rejection region
- This is because there are multiple statistics and corresponding rejection regions that are equivalent for a fixed α
- These statistics/RRs are equivalent because they all reject/fail to reject uniformly for hte same set of data

1.3 Examples

1. Let X have the pdf
$$f(x) = \begin{cases} \theta x^{\theta-1} & \text{if } 0 < x < 1 \\ 0 & \text{else} \end{cases}$$
Then, the Neyman-Pearson Lemma tells us that the most powerful α level test for testing $H_0 : \theta = 1$ vs $H_a : \theta = 2$ is of the form
$$\begin{aligned} \frac{L(1)}{L(2)} &< k \\ \text{but this test can be show to be equivalent} & \\ \text{to the test below through the following steps} & \\ \rightarrow \frac{L(1)}{L(2)} = \frac{1}{2x} &< k \\ \rightarrow 2x &> k \\ \rightarrow x &> k'(= k/2) \end{aligned}$$
So, if we want to restrict the test to have a significance of α, then we need to select k' such that $P(X > k'|\theta = 1) = \alpha$. Note
$$\begin{aligned} P(X > k'|\theta = 1) &= \int_{k'}^{1} 1\mathrm{d}x \\ &= 1 - k' \\ \text{Set equalt to } \alpha \text{ and solve for } k' & \\ \Rightarrow k' &= 1 - \alpha \end{aligned}$$

2. Let $X \sim Poisson(\lambda)$. Suppose we want to test $H_0 : \lambda = \lambda_0$ Vs. $H_a : \lambda = \lambda_a (< \lambda_0)$. Then by the Neyman-Pearson Lemma, the most powerful test for a fixed significance α rejects H_0 when
$$\frac{L(\lambda_0)}{L(\lambda_a)} < k$$
So, now we wish to find the appropriate testing statistic and corresponding Rejection region. If our test rejects when $\frac{L(\lambda_0)}{L(\lambda_a)} < k$, then our test will also reject in the

following situations:

$$\begin{aligned}
\frac{L(\lambda_0)}{L(\lambda_a)} &< k \\
\Rightarrow \frac{\frac{\lambda_0^x}{x!}e^{-\lambda_0}}{\frac{\lambda_1^x}{x!}e^{-\lambda_1}} &< k \\
\rightarrow (\frac{\lambda_0}{\lambda_1})^x e^{\lambda_1-\lambda_0} &< k \\
\rightarrow (\frac{\lambda_0}{\lambda_1})^x &< k'(=ke^{\lambda_0-\lambda_1}) \\
\rightarrow x(\ln(\lambda_0)-\ln(\lambda_1)) &< k' \\
\rightarrow x &< k''(=k'/(\ln(\lambda_0)-\ln(\lambda_1))) \\
& \quad \uparrow \text{note, } \ln(\lambda_0)-\ln(\lambda_1) > 0 \text{ since} \\
& \quad \lambda_0 > \lambda_1\text{, so the inequality does not switch}
\end{aligned}$$

So, we have show that the test that rejects when $x < k''$ rejects/fails to reject at the same places that the test which rejects when $\frac{L(\lambda_0)}{L(\lambda_a)} < k$, the most powerful test. The test which rejects $X > k''$ has a significance level of α when k is small enough such that $P(X < k''|\lambda = \lambda_0) = \alpha$.

Note: In the example above, it may not be possible to select a k'' such that $P(X < k''|\lambda = \lambda_0)$ exactly equals α. This is true in situations when the distirbution of the statistic (in this case, just X) is discrete. In these situations, the most powerful test will be defined to be the test with the largest significance **less than or equal to** α

1.4 Simple vs. Composite

- Suppose that we want to find the most powerful test when we are in a testing situation such as $H_0 : \theta = \theta_0$ Vs $H_a : \theta > \theta_0$ or $H_a : \theta < \theta_0$.
- In these cases, there is no Lemma comprable to the Neyman-Pearson Lemma that applies in this situation.
- If we consider the simple vs. simple case where the alternative is greater than (or less than) the null, then the neyman-pearson lemma CAN give us a most powerful test.
- If this most-powerful test does not depend on the value of the alternative hypothesis, then this will be the most powerful test for any alternative hypothesis value greater than (less than) θ_0
- in cases like this, the most powerful test for the simple vs. composite hypothesis test will be the test that we get for the simple vs. simple testing situation

1.5 Exercises

1. Let $X_1, ..., X_n$ be i.i.d. $N(\mu, \sigma^2)$. For all tests assume we want a test with a significance level of α
   a) Find the most powerful Test for testing $H_0 : \mu = \mu_0$ vs. $H_a : \mu > \mu_0$ when σ^2 is known, based solely on X_1
   b) Find the most powerful Test for testing $H_0 : \mu = \mu_0$ vs. $H_a : \mu > \mu_0$ when σ^2 is known, based on $X_1, ..., X_n$
   c) Find the most powerful test for testing $H_0 : \sigma = \sigma_0$ vs. $H_a : \sigma > \sigma_0$ when μ is known, based solely on X_1
   d) Find the most powerful test for testing $H_0 : \sigma = \sigma_0$ vs. $H_a : \sigma > \sigma_0$ when μ is known, based on $X_1, ..., X_n$

1.6 Solutions

1. Let $X_1, ..., X_n$ be i.i.d. $N(\mu, \sigma^2)$. For all tests assume we want a test with a significance level of α
   a) Find the most powerful Test for testing $H_0 : \mu = \mu_0$ vs. $H_a : \mu > \mu_0$ when σ^2 is known, based solely on X_1
   Soluion:
   Select $\mu_a > \mu_0$ and consider the Most Powerful test for $H_0 : \mu = \mu_0$ vs $H_a : \mu = \mu_a$. By the Neyman Pearson Lemma we know that the most powerful test is of the form

$$
\begin{aligned}
\frac{L(\mu_0)}{L(\mu_a)} &< k \\
\rightarrow \frac{\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x_1-\mu_0)^2}}{\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x_1-\mu_0)^2}} &< k \\
\rightarrow \frac{e^{-\frac{1}{2\sigma^2}(x_1-\mu_0)^2}}{e^{-\frac{1}{2\sigma^2}(x_1-\mu_a)^2}} &< k \\
\rightarrow e^{\frac{1}{2\sigma^2}(x_1-\mu_a)^2 - \frac{1}{2\sigma^2}(x_1-\mu_0)^2} &< k \\
\rightarrow \frac{1}{2\sigma^2}(x_1^2 - 2x_1\mu_a + \mu_a^2 - x_1^2 + 2x_1\mu_0 - \mu_0^2) &< k'(= \ln(k)) \\
\rightarrow 2x_1(\mu_0 - \mu_a) + \mu_a^2 - \mu_0^2 &< k''(= 2\sigma^2 k') \\
\rightarrow x_1 &> k'''(= \frac{k'' - \mu_a^2 + \mu_0^2}{2(\mu_0 - \mu_a)}) \\
&\quad \uparrow \text{Note, the inequality switches} \\
&\quad \text{because } \mu_0 < \mu_a \\
\rightarrow \frac{x_1 - \mu_0}{\sigma} &> k^{(4)}(= \frac{k''' - \mu_0}{\sigma})
\end{aligned}
$$

Now, we must solve for $k^{(4)}$. to do this we will set the probability of a type I error equal to α. This implies that

$$\begin{aligned}
\alpha = P(\text{Type I error}) &= P(\frac{X_1-\mu_0}{\sigma} > k^{(4)} | \mu = \mu_0) \\
&= P(Z > k^{(4)}) \leftarrow \text{Where } Z \sim N(0,1) \\
\Rightarrow k^{(4)} &= Z_\alpha, \text{ The } 1-\alpha \text{ percetile of the } N(0,1) \text{ distribution}
\end{aligned}$$

Therefore, the most powerful α level test for $H_0 : \mu = \mu_0$ vs $H_a : \mu = \mu_a$ rejects when $\frac{x_1-\mu_0}{\sigma} > Z_\alpha$. We note that this test does not depend on the value of μ_a. Therefore, the most powerful α level test for testing $H_0 : \mu = \mu_0$ vs. $H_a : \mu > \mu_0$ also rejects when $\frac{x_1-mu_0}{\sigma} > Z_\alpha$.

b) Find the most powerful Test for testing $H_0 : \mu = \mu_0$ vs. $H_a : \mu > \mu_0$ when σ^2 is known, based on $X_1, ..., X_n$
Soluion:

Note $L(\mu) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x_i-\mu)^2} = (\frac{1}{\sqrt{2\pi}\sigma})^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_1-\mu)^2}$ Select $\mu_a > \mu_0$ and consider the Most Powerful test for $H_0 : \mu = \mu_0$ vs $H_a : \mu = \mu_a$. By the Neyman Pearson Lemma we know that the most powerful test is of the form

$$\begin{aligned}
\frac{L(\mu_0)}{L(\mu_a)} &< k \\
\rightarrow \frac{(\frac{1}{\sqrt{2\pi}\sigma})^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu_0)^2}}{(\frac{1}{\sqrt{2\pi}\sigma})^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu_0)^2}} &< k \\
\rightarrow \frac{e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu_0)^2}}{e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu_a)^2}} &< k \\
\rightarrow e^{\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu_a)^2 - \frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu_0)^2} &< k \\
\rightarrow \frac{1}{2\sigma^2}(\sum_{i=1}^n (x_i^2 - 2x_i\mu_a + \mu_a^2) - \sum_{i=1}^n (x_i^2 - 2x_i\mu_0 + \mu_0^2)) &< k'(= \ln(k)) \\
\rightarrow \sum_{i=1}^n (2x_i(\mu_0 - \mu_a) + \mu_a^2 - \mu_0^2) &< k''(= 2\sigma^2 k') \\
\rightarrow \sum_{i=1}^n x_1 &> k'''(= \frac{k'' - n\mu_a^2 + n\mu_0^2}{2(\mu_0-\mu_a)}) \\
&\quad \uparrow \text{Note, the inequality switches} \\
&\quad \text{because } \mu_0 < \mu_a \\
\rightarrow \frac{\sum_{i=1}^n x_i - n\mu_0}{\sqrt{n}\sigma} &= \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}
\end{aligned}$$

$$> \quad k^{(4)}(= \frac{k''' - \mu_0}{\sigma})$$

Now, we must solve for $k^{(4)}$. To do this we will set the probability of a type I error equal to α. This implies that

$$\begin{aligned}
\alpha = P(\text{Type I error}) &= P(\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} > k^{(4)} | \mu = \mu_0) \\
&= P(Z > k^{(4)}) \leftarrow \text{Where } Z \sim N(0,1) \\
\Rightarrow k^{(4)} &= Z_\alpha, \text{ The } 1-\alpha \text{ percetile of the } N(0,1) \text{ distribution}
\end{aligned}$$

Therefore, the most powerful α level test for $H_0 : \mu = \mu_0$ vs $H_a : \mu = \mu_a$ rejects when $\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} > Z_\alpha$. We note that this test does not depend on the value of μ_a. Therefore, the most powerful α level test for testing $H_0 : \mu = \mu_0$ vs. $H_a : \mu > \mu_0$ also rejects when $\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} > Z_\alpha$.

c) Find the most powerful test for testing $H_0 : \sigma = \sigma_0$ vs. $H_a : \sigma > \sigma_0$ when μ is known, based solely on X_1
Soluion:
Select $\sigma_a > \sigma_0$ and consider the Most Powerful test for $H_0 : \sigma = \sigma_0$ vs $H_a : \sigma = \sigma_a$. By the Neyman Pearson Lemma we know that the most powerful test is of the form

$$\begin{aligned}
\frac{L(\sigma_0)}{L(\sigma_a)} &< k \\
\to \frac{\frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{1}{2\sigma_0^2}(x_1-\mu)^2}}{\frac{1}{\sqrt{2\pi}\sigma_a} e^{-\frac{1}{2\sigma_a^2}(x_1-\mu)^2}} &< k \\
\to \frac{\sigma_a}{\sigma_0} e^{\frac{1}{2\sigma_a^2}(x_1-\mu)^2 - \frac{1}{2\sigma_0^2}(x_1-\mu)^2} &< k \\
\to (\frac{1}{2\sigma_a^2} - \frac{1}{2\sigma_0^2})(x_1-\mu)^2 &< k'(= \ln(\frac{k\sigma_0}{\sigma_a})) \\
\to (x_1 - \mu)^2 &> k''(= k'/(\frac{1}{2\sigma_a^2} - \frac{1}{2\sigma_0^2})) \\
&\uparrow \text{note, the inequality flips since } \sigma_a > \sigma_0 \\
\to (\frac{x_1 - \mu}{\sigma_0})^2 &> k'''(k''/\sigma_0^2)
\end{aligned}$$

Now, we must solve for k'''. to do this we will set the probability of a type I error equal to α. This implies that

$$\alpha = P(\text{Type I error}) = P((\frac{x_1 - \mu}{\sigma_0})^2 > k''' | \sigma = \sigma_0)$$

$$
\begin{aligned}
&= P(\chi^2 > k''') \leftarrow \text{Where } \chi^2 \sim \chi^2_1, \text{ since } \tfrac{x_1-\mu}{\sigma_0} \sim N(0,1) \text{ when } \sigma = \sigma_0 \\
\Rightarrow k''' &= \chi^2_{1,\alpha}, \text{ The } 1-\alpha \text{ percetile of the } \chi^2_1 \text{ distribution}
\end{aligned}
$$

Therefore, the most powerful α level test for $H_0 : \sigma = \sigma_0$ vs $H_a : \sigma = \sigma_a$ rejects when $(\frac{x_1-\mu}{\sigma_0})^2 > \chi^2_{1,\alpha}$. We note that this test does not depend on the value of σ_a. Therefore, the most powerful α level test for testing $H_0 : \sigma = \sigma_0$ vs. $H_a : \sigma > \sigma_0$ also rejects when $(\frac{x_1-\mu}{\sigma_0})^2 > \chi^2_{1,\alpha}$.

d) Find the most powerful test for testing $H_0 : \sigma = \sigma_0$ vs. $H_a : \sigma > \sigma_0$ when μ is known, based on $X_1, ..., X_n$

Soluion:

Note $L(\sigma) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma^2}(x_i-\mu)^2} = (\frac{1}{\sqrt{2\pi\sigma}})^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_1-\mu)^2}$ Select $\sigma_a > \sigma_0$ and consider the Most Powerful test for $H_0 : \sigma = \sigma_0$ vs $H_a : \sigma = \sigma_a$. By the Neyman Pearson Lemma we know that the most powerful test is of the form

$$
\begin{aligned}
\frac{L(\sigma_0)}{L(\sigma_a)} &< k \\
\rightarrow \frac{(\frac{1}{\sqrt{2\pi\sigma_0}})^n e^{-\frac{1}{2\sigma_0^2}\sum_{i=1}^n (x_1-\mu)^2}}{(\frac{1}{\sqrt{2\pi\sigma_a}})^n e^{-\frac{1}{2\sigma_a^2}\sum_{i=1}^n (x_1-\mu)^2}} &< k \\
\rightarrow (\frac{\sigma_a}{\sigma_0})^n e^{\frac{1}{2\sigma_a^2}\sum_{i=1}^n (x_i-\mu)^2 - \frac{1}{2\sigma_0^2}\sum_{i=1}^n (x_i-\mu)^2} &< k \\
\rightarrow (\frac{1}{2\sigma_a^2} - \frac{1}{2\sigma_0^2})\sum_{i=1}^n (x_i-\mu)^2 &< k'(= \ln(k(\frac{\sigma_a}{\sigma_0})^n)) \\
\rightarrow \sum_{i=1}^n (x_i-\mu)^2 &> k''(= k'/(\frac{1}{2\sigma_a^2} - \frac{1}{2\sigma_0^2})) \\
&\uparrow \text{note, the inequality flips since } \sigma_a > \sigma_0 \\
\rightarrow \sum_{i=1}^n (\frac{x_i-\mu}{\sigma_0})^2 &> k'''(k''/\sigma_0^2)
\end{aligned}
$$

Now, we must solve for k'''. to do this we will set the probability of a type I error equal to α. This implies that

$$
\begin{aligned}
\alpha = P(\text{Type I error}) &= P(\sum_{i=1}^n (\frac{x_i-\mu}{\sigma_0})^2 > k''' | \sigma = \sigma_0) \\
&= P(\chi^2 > k''') \\
&\quad \uparrow \text{Where } \chi^2 \sim \chi^2_n, \text{ since } \tfrac{x_i-\mu}{\sigma_0} \sim N(0,1) \\
&\quad \text{for } i = 1, ..., n \text{ when } \sigma = \sigma_0 \\
\Rightarrow k''' &= \chi^2_{n,\alpha}, \text{ The } 1-\alpha \text{ percetile of the } \chi^2_n \text{ distribution}
\end{aligned}
$$

Therefore, the most powerful α level test for $H_0 : \sigma = \sigma_0$ vs $H_a : \sigma = \sigma_a$ rejects when $\sum_{i=1}^{n}(\frac{x_i-\mu}{\sigma_0})^2 > \chi^2_{n,\alpha}$. We note that this test does not depend on the value of σ_a. Therefore, the most powerful α level test for testing $H_0 : \sigma = \sigma_0$ vs. $H_a : \sigma > \sigma_0$ also rejects when $\sum_{i=1}^{n}(\frac{x_i-\mu}{\sigma_0})^2 > \chi^2_{n,\alpha}$.