1 Interval Estimation

Up until now we have only discussed *point estimators*, namely estimators that give a single value estimate of an unkown parameter. Now we will discuss *Interval Estimators.*

1.1 Confidence Interval

Interval estimators for an unkown parameter θ are written in the following form:

$$(\hat{\theta}_L, \hat{\theta}_U)$$

- $\hat{\theta}_L$ is the lower estimate (or lower bound)
- $\hat{\theta}_U$ is the upper estimate (upper bound)
- In general, we can make the interval larger or smaller based on the *Confidence* criterion.
  - The larger our confidence level, the higher the likelihood that the interval estimate contains the true value of θ.

  We express this notion with the following equation:

  $$P(\hat{\theta}_L < \theta < \hat{\theta}_U) = 1 - \alpha$$

  Where $1 - \alpha$ is our confidence level. (α is called the *signifigance*, we will talk more about this later).
- Because the size of the interval depends on our confidence level, we often refer to these interval estimators as *confidence intervals*

We can also construct one sided confidence intervals, which have one of the following forms:

$$(\hat{\theta}_L, \infty) \text{ or } (-\infty, \hat{\theta}_U)$$

Where

$$P(\hat{\theta}_L < \theta) = 1 - \alpha \text{ or } P(\theta < \hat{\theta}_U) = 1 - \alpha$$

- We say that intervals of the form $(\hat{\theta}_L, \infty)$ are *lower bound* confidence intervals
- We say that intervals of the form $(-\infty, \hat{\theta}_U)$ are *upper bound* confidence intervals

1.2 Pivotal Method

So, How can we construct a confidence interval? One method is the *Pivotal Method*.

Definition: A *Pivot* is a quantity with the following properties:

1. It is a function of the sample and the unkown parameter θ that is being estimated, and θ is the only unkown quanitity it depends upon
2. Its distribution does not depend on θ

Suppose Y is one of these pivots. If we know the distibution of Y, then we can find a and b such that

$$P(a < Y < b) = 1 - \alpha$$

when given an α. We then alter the inequalities inside te probability statement so that we have

$$P(\hat{\theta}_L < \theta < \hat{\theta}_U) = 1 - \alpha$$

which is the form of the confidence interval that we wanted.

1.3 Examples

1. Suppose $X \sim N(\mu, 1)$. Construct a two-sided $1 - \alpha$ confidence interval for μ.
   Solution:

$$\begin{aligned}
\rightarrow X - \mu &\sim N(0,1) \\
\Rightarrow P(Z_{1-\alpha/2} < X - \mu < Z_{\alpha/2}) &= 1 - \alpha \\
\Rightarrow P(Z_{1-\alpha/2} - X < -\mu < Z_{\alpha/2} - X) &= 1 - \alpha \\
\Rightarrow P(X - Z_{1-\alpha/2} > \mu > X - Z_{\alpha/2}) &= 1 - \alpha \\
\Rightarrow P(X - Z_{\alpha/2} < \mu < X - Z_{1-\alpha/2}) &= 1 - \alpha
\end{aligned}$$

   Where $Z_{\alpha/2}$ is the $1 - \alpha/2$ percentile of Z the standard normal distribution. Note, this means that $Z_{1-\alpha/2}$ is the $1 - (1 - \alpha/2) = \alpha/2$ percentile of Z. These are equidistant from the median of Z (which is also the mean, 0), therefore $Z_{1-\alpha/2} = -Z_{\alpha/2}$. This value could easily be looked up in a traditional Z-score table.
   So, our $1 - \alpha$ Confidence interval is $(X - Z_{\alpha/2}, X + Z_{\alpha/2})$.

2. Let $X \sim Exp(\delta)$ construct a two-sided $1 - \alpha$ Confidence interval for δ
   Solution:

$$\begin{aligned}
\rightarrow X/\delta &\sim Exp(1) \\
\Rightarrow P(e_{1-\alpha/2} < X/\delta < e_{\alpha/2}) &= 1 - \alpha \\
\Rightarrow P(e_{1-\alpha/2}/X < 1/\delta < e_{\alpha/2}/X) &= 1 - \alpha
\end{aligned}$$

$$\begin{aligned}\Rightarrow P(X/e_{1-\alpha/2} > \delta > X/e_{\alpha/2}) &= 1-\alpha \\ \Rightarrow P(X/e_{\alpha/2} < \delta < X/e_{1-\alpha/2}) &= 1-\alpha\end{aligned}$$

Where $e_{\alpha/2}$ and $e_{1-\alpha/2}$ are the $1-\alpha/2$ and $\alpha/2$ percentiles of the $Exp(1)$ distribution. These values could easily be calculated by evaluating the integrals:

$$\int_{-\infty}^{e_{1-\alpha/2}} e^{-x}\mathrm{d}x = \alpha/2 \text{ and } \int_{e_{\alpha/2}}^{\infty} e^{-x}\mathrm{d}x = \alpha/2$$

1.4 Exercises

1. Let $X \sim U(0,\theta)$
   a) Construct a lower bound $1-\alpha$ Confidence interval for θ
   b) Construct an upper bound $1-\alpha$ Confidence interval for θ
2. Let $X \sim N(\mu,\sigma^2)$ where σ^2 is assumed known, but μ is unkown. Construct a two-sided $1-\alpha$ confidence interval for μ

1.5 Solutions

1. Let $X \sim U(0,\theta)$
   a) Construct a lower bound $1-\alpha$ Confidence interval for θ
      Solution:

$$\begin{aligned}\rightarrow X/\theta &\sim U(0,1) \\ \Rightarrow P(X/\theta < u_\alpha) &= 1-\alpha \\ \Rightarrow P(1/\theta < u_\alpha/X) &= 1-\alpha \\ \Rightarrow P(\theta > X/u_\alpha) &= 1-\alpha \\ \Rightarrow P(X/u_\alpha < \theta) &= 1-\alpha\end{aligned}$$

      So our lower bound confidence interval is $(X/u_\alpha, \infty)$, where u_α is the $1-\alpha$ percentile of $U(0,1)$
   b) Construct an upper bound $1-\alpha$ Confidence interval for θ
      Solution:

$$\begin{aligned}\rightarrow X/\theta &\sim U(0,1) \\ \Rightarrow P(X/\theta > u_{1-\alpha}) &= 1-\alpha \\ \Rightarrow P(1/\theta > u_{1-\alpha}/X) &= 1-\alpha \\ \Rightarrow P(\theta < X/u_{1-\alpha}) &= 1-\alpha\end{aligned}$$

So our upper bound confidence interval is $(0, X/u_{1-\alpha})$, where $u_{1-\alpha}$ is the α percentile of $U(0,1)$

2. Let $X \sim N(\mu, \sigma^2)$ where σ^2 is assumed known, but μ is unkown. Construct a two-sided $1-\alpha$ confidence interval for μ
Solution:

$$\begin{aligned}
\rightarrow \frac{X-\mu}{\sigma} &\sim N(0,1) \\
\Rightarrow P(Z_{1-\alpha/2} < \frac{X-\mu}{\sigma} < Z_{\alpha/2}) &= 1-\alpha \\
\Rightarrow P(\sigma Z_{1-\alpha/2} < X-\mu < \sigma Z_{\alpha/2}) &= 1-\alpha \\
\Rightarrow P(\sigma Z_{1-\alpha/2} - X < -\mu < \sigma Z_{\alpha/2} - X) &= 1-\alpha \\
\Rightarrow P(X - \sigma Z_{1-\alpha/2} > \mu > X - \sigma Z_{\alpha/2}) &= 1-\alpha \\
\Rightarrow P(X - \sigma Z_{\alpha/2} < \mu < X - \sigma Z_{1-\alpha/2}) &= 1-\alpha
\end{aligned}$$

Where $Z_{\alpha/2}$ and $Z_{1-\alpha/2}$ are the $1-\alpha/2$ and the $\alpha/2$ percentiles of Z, respectively. Thus our two-sided confidence interval is $(X - \sigma Z_{\alpha/2}, X + \sigma Z_{\alpha/2})$

2 Large Sample Interval Estimation

When dealing with large samples ($n > 50$ in this class), remember that the CLT gives us

$$U_n = \frac{\bar{X} - E[X_1]}{\sqrt{V[X_1]/n}}$$

is *Approximately* distributed $N(0,1)$. This means that we can construct *approximate* pivots in the form of U_n. Since the distrubtion of U_n is approximately $N(0,1)$, that means that

$$\begin{aligned}
\rightarrow P(-Z_{\alpha/2} < U_n < Z_{\alpha/2}) &\approx 1-\alpha \\
\rightarrow P(U_n < Z_\alpha) &\approx 1-\alpha \\
\rightarrow P(-Z_\alpha < U_n) &\approx 1-\alpha
\end{aligned}$$

Which gives us our two-sided and one-sided confidence intervals.

$\rightarrow$ In these problems, we often have have to estimate $V[X_1]$ using the sample variance to derive our final CI for $E[X_1] = \mu$, but not just any estimator will work (i.e. CLT theorem will not always hold for any estimatro of $V[X_1]$).

Theorem 1 (CLT & Consistent variance Estimators). *If an estimator $\widehat{V[X_1]}$ is a consistent esimator for $V[X_1]$, then for $U_n = \frac{\bar{X}-E[X_1]}{\sqrt{\widehat{V[X_1]}/n}}$*

$$\lim_{n\to\infty} F_{U_n}(u) = \lim_{n\to\infty} P(U_n < u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \mathrm{d}t \text{ for all } u$$

or in otherwords, for sufficiently large n,

$$U_n = \frac{\bar{X} - E[X_1]}{\sqrt{\widehat{V[X_1]}/n}} \overset{approx}{\sim} N(0,1)$$

Finding a consistent estimator for $V[X_1]$ can b tricky. Luckily, two estimators we have encounters are (for the problems in this course) always consistent for $V[X_1]$.

Theorem 2 (Consistency of Sample Variance and MLE)**.** *Let $X_1, \dots, X_n$ be iid with some distribution. Then the sample variance, $S^2 = \sum_{i=1}^n (X_i - \overline{X})^2/(n-1)$ (where $\overline{X} = \sum_{i=1}^n X_i/n$) and the MLE of $V[X_1]$ are both consistent esimators for $V[X_1]$*

2.1 Example

1. Let $X_1, ..., X_{100}$ be i.i.d with an unkown distribution with $\bar{X} = 10$ and $S_X^2 = 25$. Construct an *approximate* $1-\alpha$ two sided CI for μ the true population average. *Solution:*

$$\begin{aligned}
U_{100} = \frac{\bar{X} - \mu}{\sqrt{V[X_1]/100}} &\sim N(0,1)(\text{Approximately}) \\
\Rightarrow P(-Z_{\alpha/2} < U_{100} < Z_{\alpha/2}) &\approx 1-\alpha \\
\Rightarrow P(-Z_{\alpha/2} < \frac{\bar{X} - \mu}{\sqrt{V[X_1]/100}} < Z_{\alpha/2}) &\approx 1-\alpha \\
\Rightarrow P(-Z_{\alpha/2}\sqrt{V[X_1]/100} < \bar{X} - \mu < Z_{\alpha/2}\sqrt{V[X_1]/100}) &\approx 1-\alpha \\
\Rightarrow P(-Z_{\alpha/2}\sqrt{V[X_1]/100} - \bar{X} < -\mu < Z_{\alpha/2}\sqrt{V[X_1]/100} - \bar{X}) &\approx 1-\alpha \\
\Rightarrow P(\bar{X} - Z_{\alpha/2}\sqrt{V[X_1]/100} < \mu < \bar{X} + Z_{\alpha/2}\sqrt{V[X_1]/100}) &\approx 1-\alpha
\end{aligned}$$

So, this gives us a $1-\alpha$ CI of $(\bar{X} - Z_{\alpha/2}\sqrt{V[X_1]/100}, \bar{X} + Z_{\alpha/2}\sqrt{V[X_1]/100})$, but we only have $\bar{X}$, and not $V[X_1]$. Because we are constructing an approximate CI, we generally consider it reasonable to estimate $V[X_1]$ with S_X^2. So, our final approximate CI will be :

$$(10 - \frac{1}{2}Z_{\alpha/2}, 10 + \frac{1}{2}Z_{\alpha/2})$$

2.2 Exercises

Let $X_1, ..., X_{100}$ be i.i.d. *Bernoulli*(p). From this sample we have a sum of $\sum_{i=1}^{100} X_i = 49$. Construct approximate $1-\alpha$ upper bound CI.

a) First, Start with $P(-Z_\alpha < \frac{\bar{X}-p}{\sqrt{V[X_1]/100}}) = 1-\alpha$ and construct a $1-\alpha$ upper-bound that is a function of $\bar{X}$ and $V[X_1]$

b) Then, substitue the MLE, $\widehat{V[X_1]}$ into your formula for $V[X_1]$ to get you CI

2.3 Solutions

Let $X_1, ..., X_{100}$ be i.i.d. $Bernoulli(p)$. From this sample we have a sum of $\sum_{i=1}^{100} X_i = 49$. Construct approximate $1-\alpha$ upper bound CI.

a) First, Start with $P(-Z_\alpha < \frac{\bar{X}-p}{\sqrt{V[X_1]/100}}) = 1-\alpha$ and construct a $1-\alpha$ upper-bound that is a function of $\bar{X}$ and $V[X_1]$
Solution:

$$\begin{aligned}
\frac{\bar{X}-p}{\sqrt{V[X_1]/100}} &\sim N(0,1)(\text{Aprroximately}) \\
\Rightarrow P(-Z_\alpha < \frac{\bar{X}-p}{\sqrt{V[X_1]/100}}) &\approx 1-\alpha \\
\Rightarrow P(-Z_\alpha\sqrt{V[X_1]/100} < \bar{X}-p) &\approx 1-\alpha \\
\Rightarrow P(-Z_\alpha\sqrt{V[X_1]/100} - \bar{X} < -p) &\approx 1-\alpha \\
\Rightarrow P(p < \bar{X} + Z_\alpha\sqrt{V[X_1]/100}) &\approx 1-\alpha
\end{aligned}$$

So, our $1-\alpha$ upper bound CI is of the form $(-\infty, \bar{X} + Z_\alpha\sqrt{V[X_1]/100})$

b) Then, substitue the MLE, $\widehat{V[X_1]}$ into your formula for $V[X_1]$ to get you CI
Solution:
The MLE for p in this situation is $\overline{X} = \frac{1}{n}\sum_{i=1}^{100} X_i$, so, by the invariance property of MLE We have that the MLE for $V[X1]$ is $\widehat{V[X_1]} = \overline{X}(1-\overline{X})$. So, since $\overline{X} = \frac{49}{100}$, our final CI is

$$(-\infty, \frac{49}{100} + Z_\alpha\sqrt{\frac{49}{100}(1-\frac{49}{100})/100})$$