Homework 3

Solutions

1. Let $X_1, X_2, ..., X_{121}$ be i.i.d. $\Gamma(2, \beta)$ where $\beta > 0$. Construct a statistic that is approximately distributed $N(0, 1)$

   Solution: We know that by the Central Limit Theorem

   $$\frac{\bar{X} - E[X_1]}{\sqrt{V[X_1]/n}} = \frac{\bar{X} - 2\beta}{\sqrt{2\beta^2/121}} = 11\frac{\bar{X} - 2\beta}{\sqrt{2\beta^2}}$$

   Is *approximately* normally distributed with a mean of 0, and a standard deviation of 1.

2. Let $X_1, X_2, ..., X_{144}$ be i.i.d. $Beta(\alpha, \beta)$ where $\alpha > 0$ and $\beta > 0$. Construct a statistic that is approximately distributed $N(0, 1)$

   Solution: We know that by the Central Limit Theorem

   $$\frac{\bar{X} - E[X_1]}{\sqrt{V[X_1]/n}} = \frac{\bar{X} - \frac{\alpha}{\alpha+\beta}}{\sqrt{\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}/144}} = 12\frac{\bar{X} - \frac{\alpha}{\alpha+\beta}}{\sqrt{\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}}}$$

   Is *approximately* normally distributed with a mean of 0, and a standard deviation of 1.

3. Let $X_1, X_2, ..., X_{100}$ be i.i.d. $Uniform(0, \theta)$, where $\theta > 0$. Construct a statistic that is approximately distributed $N(0, 1)$

   Solution: We know that by the Central Limit Theorem

   $$\frac{\bar{X} - E[X_1]}{\sqrt{V[X_1]/n}} = \frac{\bar{X} - \frac{\theta}{2}}{\sqrt{\theta^2/1200}} = 20\sqrt{3}\frac{\bar{X} - \frac{\theta}{2}}{\theta}$$

   Is *approximately* normally distributed with a mean of 0, and a standard deviation of 1.

4. Let $X_1, X_2, ..., X_{100}$ be i.i.d. $Uniform(-\theta, \theta)$, where $\theta > 0$. Construct a statistic that is approximately distributed $N(0, 1)$

Solution: We know that by the Central Limit Theorem

$$\frac{\bar{X} - E[X_1]}{\sqrt{V[X_1]/n}} = \frac{\bar{X}}{\sqrt{4\theta^2/1200}} = 10\sqrt{3}\frac{\bar{X}}{\theta}$$

Is *approximately* normally distributed with a mean of 0, and a standard deviation of 1.

5. Let $X_1, X_2, ..., X_{100}$ be i.i.d. $Exp(\delta)$. Show why

$$U = (\sum_{i=1}^{n} \frac{X_i}{10\delta}) - 10$$

is *approximately* distributed $N(0,1)$.

Solution:

$$\begin{aligned}
U &= (\sum_{i=1}^{100} \frac{X_i}{10\delta}) - 10 \\
&= (\sum_{i=1}^{100} \frac{X_i}{10\delta}) - \frac{100\delta}{10\delta} \\
&= \frac{1}{10\delta}[(\sum_{i=1}^{100} X_i) - 100\delta] \\
&= \frac{(\sum_{i=1}^{100} X_i) - 100\delta}{10\delta} \\
&= 100(\frac{\bar{X} - \delta}{10\delta}) \\
&= \frac{\bar{X} - \delta}{\delta/10} \\
&= \frac{\bar{X} - \delta}{\sqrt{\delta^2/100}} \\
&= \frac{\bar{X} - E[X_1]}{\sqrt{V[X_1]/100}}
\end{aligned}$$

So, by the CLT, U is *approximately* distributed $N(0,1)$

6. Let $X_1, ..., X_{100}$ be i.i.d. $Bern(.5)$.

   a) Show why

$$U = \frac{2}{10}(\sum_{i=1}^{100} X_i) - 10$$

   is approximately normally distributed.
   Solution:

First, we note

$$\begin{aligned} X_1, ..., X_{100} &\sim Bern(.5) \\ \Rightarrow E[x_1] &= 1/2 \\ \Rightarrow V[X_1] &= (1/2)(1-1/2) \\ &= (1/2)^2 \end{aligned}$$

Thus,

$$\begin{aligned} U &= \frac{2}{10}(\sum_{i=1}^{100} X_i) - 10 \\ &= \frac{20}{100}(\sum_{i=1}^{100} X_i) - 10 \\ &= 20\bar{X} - 10 \\ &= \frac{10\bar{X} - 5}{1/2} \\ &= \frac{\bar{X} - 1/2}{\sqrt{(1/2)^2/100}} \\ &= \frac{\bar{X} - E[X_1]}{\sqrt{V[X_1]/100}} \end{aligned}$$

So, by CLT U is approximately distributed $N(0,1)$

b) Approximate $P(U < 0)$

Solution:

$$\begin{aligned} P(U < 0) &\approx P(Z < 0) \leftarrow \text{From part a, where } Z \sim N(0,1) \\ &= \frac{1}{2} \leftarrow \text{ Since } Z \text{ is symmetric about } 0 \end{aligned}$$

7. Problem 7.42 from the book (p 374)

Solution:

Let $\overline{Y}$ denote the sample mean strength of 100 random selected pieces of glass. Thus, the quantity $(\overline{Y} - 14.5)/.2$ has an approximate standard normal distribution.

a. $P(\overline{Y} > 14) \approx P(Z > 2.5) = .0062$.

b. We have that $P(-1.96 < Z < 1.96) = .95$. So, denoting the required interval as (a, b) such that $P(a < \overline{Y} < b) = .95$, we have that $-1.96 = (a - 14)/.2$ and $1.96 = (b - 14)/.2$. Thus, $a = 13.608$, $b = 14.392$.

8. Problem 7.52 from the book (p 375)

   Solution:

   Let $\overline{Y}$ denote the average resistance for the 25 resistors. With $\mu = 200$ and $\sigma = 10$ ohms,

   a. $P(199 \le \overline{Y} \le 202) \approx P(-.5 \le Z \le 1) = .5328$.

   b. Let X = *total resistance* of the 25 resistors. Then,

$$P(X \le 5100) = \mathrm{P}(\overline{Y} \le 204) \approx P(Z \le 2) = .9772.$$

Challenge Question:

Our presentation of the CLT stated that given $X_1, ..., X_n$ i.i.d. with $E[X_i] = \mu$ and $V[X_i] = \sigma^2$ then

$$U_n = \frac{\bar{X} - E[X_1]}{\sqrt{V[X_1]/n}}$$

as a sequence of random variables converges to the standard normal distribution. That is:

$$\lim_{n\to\infty} F_{U_n}(u) = \lim_{n\to\infty} P(U_n < u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \mathrm{d}t \text{ for all } u$$

Using this, find the distribution that the sequence of random variables, $a(U_n - b)$, where a(> 0) and b are real numbers, converges to.