1 Hypothesis Testing

1.1 Hypotheses

Comparing two possible states of reality.

- One is typically the state that is widely accepted. This is often called the *Null Hypothesis*
- The other is the opposing, or alternative state that contradicts the null hypothesis. This is refered to as the *Alternative Hypothesis*

1.2 Examples

1. It is generally believed that when we toss a coin in the air, the probability that the coin lands heads up is $\frac{1}{2}$.

   I personally am a sceptic. I believe that coins actually favor heads, and come up heads with a probability of 0.9

   Here, the null hypothesis would be $H_0 : p = .5$

   And the alternative Hypothesis would be $H_a : p = .9$

   These Hypotheses would constitute a simple Vs. Simple hypothesis testing situation

2. My professors tell me that I get an average of 2 homework assignments a week, but I swear that I get more than two assignments a week on average. We will assume that the number of homework assignments I get ina given week follows a poisson distirbution.

   Here, $H_0 : \lambda = 2$ and $H_a : \lambda > 2$ These hypotheses would constitute a simple vs. composite testing situation

1.3 Rejection Regions

- Once we have our hypotheses, we can then proceed to collect data to help us conclude that one hypothesis is correct
- To do this, we take the data we collect and construct a *test statistic*
- If that statistic falls into a pre-determined *Rejection Region*, then we reject the null hypothesis in favor of the alternative.
- If our statistic doesn't fall within the rejection region, then we fail to reject the null hypothesis

1.4 Examples

1. Consider our first example, and suppose we flip a single coin. If we consider our hypotheses, and let our test static just be the single obeservation then one possible rejection region we might use would be

   $$RR = \{1\}$$

   Also, denoted just $\{x = 1\}$

   So, when we observe $X = x$, if $x = 0$, then we would fail to reject the null hypothesis, an if $x = 1$ then we would reject the null hypothesis in favor of the alternative hypothesis.

2. Consider our second example with my homework. Suppose we observe $X = x$. An example of a rejection region we could use is

   $$RR = \{x : x \geq 3\}$$

   Also, denoted just $\{x \geq 3\}$

   So, if $x = 0, 1, 2$ then we would fail to reject the null hypothesis, and if $x \geq 3$, then we would reject the null hypothesis

1.5 Errors

- When doing hypothesis testing there are 4 possible outcomes of your test
  1. The null hypothesis is true, and your test results in you failing to reject the null hypothesis
  2. the null hypothesis is true, and your test results in you rejecting the null hypothesis

     $\rightarrow$ This is called a *Type I Error*
  3. The alternative Hypothesis is true, and you reject the null hypothesis in favor of the alternative
  4. The alternative Hypothesis is true, but you fail to reject the null hypothesis in favor of the alternative

     $\rightarrow$ This is called a *Type II Error*

- This means that the probability of a type I error (denoted α, and often refered to as the *significance*) is the probability that the test staticic falls into the rejection region when the null hypothesis is true

- Additionally, the probability of a type II error (denoted β) is the probability that the test statistic doesn't fall into the rejection region when the alternative hypothesis is true.

1.6 Examples

1. Return again to our first example with the coin. Fine the probability of a Type I and Type II error using the RR $\{x = 1\}$
   Solution:

$$\begin{aligned}
P(\text{Type I Error}) = \alpha &= P(X \in RR|\text{Null Hypothesis is true}) \\
&= P(X \in \{x = 1\}|p = .5) \\
&= P(X = 1|p = .5) \\
&= .5 \\
P(\text{Type II Error}) = \beta &= P(X \notin RR|\text{Alternative Hypothesis is true}) \\
&= P(X \notin \{x = 1\}|p = .9) \\
&= P(X = 0|p = .9) \\
&= .1
\end{aligned}$$

2. Consider my homework problem again. Using the same rejection region ($\{x \geq 3\}$), find the probability of a type I and type II error using the RR $\{x \geq 3\}$. To find the type II error, since the alternative hypothesis is a range of values, find the probability of a type II error in terms of $\lambda = c > 2$

$$\begin{aligned}
P(\text{Type I Error}) = \alpha &= P(X \in RR|\text{Null Hypothesis is true}) \\
&= P(X \in \{x \geq 3\}|\lambda = 2) \\
&= 1 - P(X \notin \{x \geq 3\}|\lambda = 2) \\
&= 1 - (P(X = 0|\lambda = 2) + P(X = 1|\lambda = 2) + P(X = 2|\lambda = 2)) \\
&= 1 - \sum_{i=1}^{2} \frac{2^i}{i!} e^{-2} \\
P(\text{Type II Error}) = \beta &= P(X \notin RR|\text{Alternative Hypothesis is true}) \\
&= P(X \notin \{x \geq 3\}|\lambda = c) \\
&= P(X = 0|\lambda = c) + P(X = 1|\lambda = c) + P(X = 2|\lambda = c) \\
&= \sum_{i=0}^{2} \frac{c^i}{i!} e^{-c}
\end{aligned}$$

1.7 fixing α

Lets consider a hypothesis testing situation and two possible rejection regions, RR an RR' such that RR'$\subset$RR. Let α' be the probability of a type I error using RR', and let α be the probability of a type I error using RR. Let $U = u$ be our test statistic.

we see that

$$\begin{aligned}
\alpha' &= P(U \in RR' | H_0 \text{ is true}) \\
&\leq P(U \in RR | H_0 \text{ is true}) \\
&= \alpha
\end{aligned}$$

So, as we increase the size of our rejection region we expect our probability of a type I error to increase as well. This makes sense, because as we increase the size of our rejection region, we are increaing the probability of rejecting the null hypothesis regardless of what the true state is.

Additionally, if we let β be the probability of a type II error using RR, and let β' be the proabbility of a type II error using RR', then we see that

$$\begin{aligned}
\beta' &= P(U \notin RR' | H_a \text{ is true}) \\
&\geq P(U \notin RR | H_a \text{ is true}) \\
&= \beta
\end{aligned}$$

So, as the size of the rejection region increases, the proabbility of a type II error decreases.

- If we try to adjust α or β, this will affet our rejection region, which in trun will affter the proabbility of the other type of error
- So, traditionally, we fix our probability of a type I error because a type I error tends to be associatd with more serious penalties. Based on this fixed level of α we determine what our rejection region should be.

1.8 Exercises

1. Consider the coin flipping again, but lets suppose that we observe two coins, and our test statistic is X_1+X_2, where X_1 and X_2 are the results from each coin. Find the probability of a type I error and a type II error when the rejection region is $RR_1 = \{x \geq 1\}$ and when it is $RR_2\{x \geq 2\}$
2. Suppose that we observe two independent weeks worht of homework from the second example. We will call them X_1 and X_2. If our test statistic is $X_1 + X_2$, find the probability of making a type I error and a type II error (using $\lambda = c > 2$) when the rejection region is $RR_1 = \{x \geq 3\}$ and when the rejection region is $RR_2 = \{x \geq 4\}$

1.9 Solutions

1. Consider the coin flipping again, but lets suppose that we observe two coins, and our test statistic is X_1+X_2, where X_1 and X_2 are the results from each coin. Find the probability of a type I error and a type II error when the rejection region is

$RR_1 = \{x \geq 1\}$ and when it is $RR_2\{x \geq 2\}$
Solution:

a) $RR_1 = \{x \geq 1\}$

$$\begin{aligned} P(\text{Type I Error}) = \alpha &= P(U \in RR_1|\text{Null Hypothesis is true}) \\ &= P(U \in \{x \geq 1\}|p = .5) \\ &= P(U = 1|p = .5) + P(U = 2|p = .5) \\ &= .5 + .25 \\ &= .75 \\ P(\text{Type II Error}) = \beta &= P(U \notin RR_1|\text{Alternative Hypothesis is true}) \\ &= P(U \notin \{x \geq 1\}|p = .9) \\ &= P(U = 0|p = .9) \\ &= (.1)^2 \end{aligned}$$

b) $RR_2 = \{x \geq 2\}$

$$\begin{aligned} P(\text{Type I Error}) = \alpha &= P(U \in RR_2|\text{Null Hypothesis is true}) \\ &= P(U \in \{x \geq 2\}|p = .5) \\ &= P(U = 2|p = .5) \\ &= .25 \\ P(\text{Type II Error}) = \beta &= P(U \notin RR_2|\text{Alternative Hypothesis is true}) \\ &= P(U \notin \{x \geq 2\}|p = .9) \\ &= P(U = 0|p = .9) + P(U = 1|p = .9) \\ &= (.1)^2 + 2(.1)(.9) \end{aligned}$$

2. Suppose that we observe two independent weeks worht of homework from the second example. We will call them X_1 and X_2. If our test statistic is $X_1 + X_2$, find the probability of making a type I error and a type II error (using $\lambda = c$) when the rejection region is $RR_1 = \{x \geq 3\}$ and when the rejection region is $RR_2 = \{x \geq 4\}$
Solutions:

a) $RR_1 = \{x \geq 3\}$

$$\begin{aligned} P(\text{Type I Error}) = \alpha &= P(U \in RR_1|\text{Null Hypothesis is true}) \\ &= P(U \in \{x \geq 3\}|\lambda = 2) \\ &= 1 - P(U \notin \{x \geq 3\}|\lambda = 2) \end{aligned}$$

$$
\begin{aligned}
&= 1-(P(U=0|\lambda=2)+P(U=1|\lambda=2)+P(U=2|\lambda=2)) \\
&= 1-\sum_{i=0}^{2}\frac{4^i}{i!}e^{-4} \\
P(\text{Type II Error})=\beta &= P(U\notin RR_1|\text{Alternative Hypothesis is true}) \\
&= P(U\notin\{x\geq 3\}|\lambda=c) \\
&= P(U=0|\lambda=c)+P(U=1|\lambda=c)+P(U=2|\lambda=c) \\
&= \sum_{i=0}^{2}\frac{(2c)^i}{i!}e^{-2c}
\end{aligned}
$$

b) $RR_2=\{x\geq 4\}$

$$
\begin{aligned}
P(\text{Type I Error})=\alpha &= P(U\in RR_2|\text{Null Hypothesis is true}) \\
&= P(U\in\{x\geq 4\}|\lambda=c) \\
&= 1-P(U\notin\{x\geq 4\}|\lambda=c) \\
&= 1-[P(U=0|\lambda=2)+P(U=1|\lambda=2) \\
&\qquad\qquad +P(U=2|\lambda=2)+P(U=3|\lambda=2)] \\
&= 1-\sum_{i=0}^{3}\frac{4^i}{i!}e^{-4} \\
P(\text{Type II Error})=\beta &= P(U\notin RR_2|\text{Alternative Hypothesis is true}) \\
&= P(U\notin\{x\geq 4\}|\lambda=c) \\
&= P(U=0|\lambda=c)+P(U=1|\lambda=c) \\
&\qquad\qquad +P(U=2|\lambda=c)+P(U=3|\lambda=c) \\
&= \sum_{i=0}^{3}\frac{(2c)^i}{i!}e^{-2c}
\end{aligned}
$$