HW 1

Solutions

1. Let X be distributed Gamma(α, β) and let $a > 0$ be a constant. Find the distribution of $U = aX$.
   Solution:

$$\begin{aligned} & h(X) = aX \\ \Rightarrow\ & h^{-1}(U) = U/a \\ \Rightarrow\ & f_U(u) = f_X(h^{-1}(u))|\frac{\mathrm{d}}{\mathrm{d}u}h^{-1}(u)| \end{aligned}$$

$$\begin{aligned} f_X(h^{-1}(u))|\frac{\mathrm{d}}{\mathrm{d}u}h^{-1}(u)| &= \frac{1}{\Gamma(\alpha)\beta^\alpha}(u/a)^{\alpha-1}e^{-\frac{(u/a)}{\beta}}|\frac{1}{a}| \\ &= \frac{1}{\Gamma(\alpha)\beta^\alpha}u^{\alpha-1}(\frac{1}{a})^\alpha e^{-\frac{u}{a\beta}} \\ &= \frac{1}{\Gamma(\alpha)(a\beta)^\alpha}u^{\alpha-1}e^{-\frac{u}{a\beta}} \end{aligned}$$

   Where $0 < u < \infty$.
   Thus $U \sim \Gamma(\alpha, a\beta)$

2. Let X be distributed Normal(μ, σ^2) and let $a > 0, b$ be constants. Find the distribution of $U = a(X + b)$.
   Solution:

$$\begin{aligned} & h(X) = a(X+b) \\ \Rightarrow\ & h^{-1}(U) = \frac{U - ab}{a} \\ \Rightarrow\ & f_U(u) = f_X(h^{-1}(u))|\frac{\mathrm{d}}{\mathrm{d}u}h^{-1}(u)| \end{aligned}$$

$$f_X(h^{-1}(u))|\frac{\mathrm{d}}{\mathrm{d}u}h^{-1}(u)| = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}(\frac{(U-ab)/a-\mu}{\sigma})^2}|\frac{1}{a}|$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{(U-ab)/a-\frac{a\mu}{a}}{\sigma})^2} \frac{1}{a} \\
&= \frac{1}{\sqrt{2\pi}a\sigma} e^{-\frac{1}{2}(\frac{U-(a(b+\mu))}{a\sigma})^2}
\end{aligned}
$$

Where $-\infty < u < \infty$.
Thus $U \sim N(a(b+\mu), (a\sigma)^2)$

3. Let X be distributed Uniform$(0, \theta)$ where $\theta > 0$ and let $a > 0$ be a constant. Find the distribution of $U = aX$.
Solution:

$$
\begin{aligned}
F_U(u) &= P(U \leq u) \\
&= P(aX \leq u) \\
&= P(X \leq \frac{u}{a}) \\
&= \int_0^{u/a} \frac{1}{\theta} \mathrm{d}x \ (= F_X(u/a)) \\
&= \frac{u}{a\theta}
\end{aligned}
$$

Where $0 < u/a < \theta \Rightarrow 0 < u < a\theta$.

$$
\begin{aligned}
\Rightarrow f_U(u) &= \frac{\mathrm{d}}{\mathrm{d}u} F_U(u) \\
&= \frac{\mathrm{d}}{\mathrm{d}u} \frac{u}{a\theta} \\
&= \frac{1}{a\theta}
\end{aligned}
$$

Where $0 < u < a\theta$.
Thus $U \sim U(0, a\theta)$

4. Let X_1, X_2 be distributed Bernoulli(p), and let X_1 and X_2 be independent. Find the distribution of $U = X_1 + X_2$
Solution:

$$X_1, X_2 \sim \text{Bernoulli}(p) \Rightarrow m_{X_1}(t) = m_{X_2}(t) = 1 - p + pe^t$$

$$
\begin{aligned}
M_U(t) &= E[e^{Ut}] \\
&= E[e^{(X_1+X_2)t}] \\
&= E[e^{X_1 t} e^{X_2 t}] \\
&= E[e^{X_1 t}] E[e^{X_2 t}] \leftarrow \text{since } X_1 \text{ and } X_2 \text{ are independent} \\
&= (1 - p + pe^t)(1 - p + pe^t) \\
&= (1 - p + pe^t)^2
\end{aligned}
$$

Thus $U \sim \text{Binomial}(2, p)$

5. Let X_1, X_2 be distributed Poisson(λ), and let X_1 and X_2 be independent. Find the distribution of $U = X_1 + X_2$
 Solution:

$$X_1, X_2 \sim \text{Poisson}(\lambda) \Rightarrow m_{X_1}(t) = m_{X_2}(t) = e^{\lambda(e^t-1)}$$

$$\begin{aligned} M_U(t) &= E[e^{Ut}] \\ &= E[e^{(X_1+X_2)t}] \\ &= E[e^{X_1t}e^{X_2t}] \\ &= E[e^{X_1t}]E[e^{X_2t}] \leftarrow \text{since } X_1 \text{ and } X_2 \text{ are independent} \\ &= e^{\lambda(e^t-1)}e^{\lambda(e^t-1)} \\ &= e^{(2\lambda)(e^t-1)} \end{aligned}$$

Thus $U \sim \text{Poisson}(2\lambda)$

6. Let X_1, X_2 be distributed Gamma(α, β), and let X_1 and X_2 be independent. Find the distribution of $U = X_1 + X_2$
 Solution:

$$X_1, X_2 \sim \text{Gamma}(\alpha, \beta) \Rightarrow m_{X_1}(t) = m_{X_2}(t) = (1 - \beta t)^{-\alpha} \text{ for } t < \tfrac{1}{\beta}$$

$$\begin{aligned} M_U(t) &= E[e^{Ut}] \\ &= E[e^{(X_1+X_2)t}] \\ &= E[e^{X_1t}e^{X_2t}] \\ &= E[e^{X_1t}]E[e^{X_2t}] \leftarrow \text{since } X_1 \text{ and } X_2 \text{ are independent} \\ &= (1-\beta t)^{-\alpha}(1-\beta t)^{-\alpha} \\ &= (1-\beta t)^{-(2\alpha)}, \text{ when } t < \tfrac{1}{\beta} \end{aligned}$$

Thus $U \sim \text{Gamma}(2\alpha, \beta)$

7. Let X be distributed Uniform$(0, 1)$. Find the distribution of $U = -ln(X)$
 Solution:

$$\begin{aligned} F_U(u) &= P(U \leq u) \\ &= P(-ln(X) \leq u) \\ &= P(X > e^{-u}) \\ &= \int_{e^{-u}}^{1} \frac{1}{1} \mathrm{d}x \\ &= 1 - e^{-u} \end{aligned}$$

Where

$$\begin{aligned} 0 < x < 1 &\Rightarrow -\infty < \ln(x) < 0 \\ &\Rightarrow 0 < -ln(x) < \infty \\ &\Rightarrow 0 < u < \infty \end{aligned}$$

$$\begin{aligned} \Rightarrow f_U(u) &= \frac{\mathrm{d}}{\mathrm{d}u} F_U(u) \\ &= \frac{\mathrm{d}}{\mathrm{d}u} 1 - e^{-u} \\ &= e^{-u} \end{aligned}$$

Where $0 < u < \infty$.
Thus $U \sim$ Exponential(1)

8. Let $X_1, X_2, ..., X_n$ all be be distributed Normal(μ, σ^2). Let $X_1, X_2, ..., X_n$ be mutually independent.

   a) Find the distribution of $U = \sum_{i=1}^{m} X_i$ for positive integer $m \leq n$
   Solution:

$$X_1, X_2, ..X_n \sim \text{Normal}(\mu, \sigma) \Rightarrow m_{X_1}(t) = m_{X_2}(t) = ... = m_{X_m}(t) = e^{\mu t + \sigma^2 \frac{t^2}{2}}$$

$$\begin{aligned} M_U(t) &= E[e^{Ut}] \\ &= E[e^{(\sum_{i=1}^{m} X_i)t}] \\ &= E[e^{(\sum_{i=1}^{m} X_i t)}] \\ &= E[\prod_{i=1}^{m} e^{X_i t}] \\ &= \prod_{i=1}^{m} E[e^{X_i t}] \leftarrow \text{since } X_1, X_2, ..., X_m \text{ are mutually independent} \\ &= \prod_{i=1}^{m} e^{\mu t + \sigma^2 \frac{t^2}{2}} \\ &= e^{(m\mu)t + (m\sigma^2)\frac{t^2}{2}} \end{aligned}$$

   Thus $U \sim$ Normal$(m\mu, m\sigma^2)$

   b) Find the distribution of Z^2 where $Z = \frac{X_1 - \mu}{\sigma}$ *Hint:* Can the solution from problem #2 be applied here for specific values of a and b?
   Solution:

$$
\begin{aligned}
M_{Z^2}(t) &= E[e^{Z^2 t}] \\
&= \int_{-\infty}^{\infty} \frac{e^{z^2 t}}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \mathrm{d}z \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1-2t}{2} z^2} \mathrm{d}z \\
&= \frac{1}{\sqrt{1-2t}} \int_{-\infty}^{\infty} \frac{\sqrt{1-2t}}{\sqrt{2\pi}} e^{-\frac{1-2t}{2} z^2} \mathrm{d}z \rightarrow \text{ assuming } 2t < 1 \\
&= \frac{1}{(1-2t)^{1/2}}
\end{aligned}
$$

This is the mgf of $\Gamma(\frac{1}{2}, 2)$. Therefore $Z^2 \sim \Gamma(\frac{1}{2}, 2)$.

9. Problem 6.14 from the book (p 309)

   Since Y_1 and Y_2 are independent, so $f(y_1, y_2) = 18(y_1 - y_1^2)y_2^2$, for $0 \le y_1 \le 1$, $0 \le y_2 \le 1$.
   Let $U = Y_1 Y_2$. Then,

   $F_U(u) = P(U \le u) = P(Y_1 Y_2 \le u) = P(Y_1 \le u/Y_2) = P(Y_1 > u/Y_2) = 1 - \int_u^1 \int_{u/y_2}^1 18(y_1 - y_1^2) y_2^2 dy_1 dy_2$

   $= 9u^2 - 8u^3 + 6u^3 \ln u.$

   $f_U(u) = F_U'(u) = 18u(1 - u + u\ln u)$, $0 \le u \le 1$.

10. Problem 6.15 from the book (p 309)

    Let U have a uniform distribution on (0, 1). The distribution function for U is $F_U(u) = P(U \le u) = u$, $0 \le u \le 1$. For a function G, we require $G(U) = Y$ where Y has distribution function $F_Y(y) = 1 - e^{-y^2}$, $y \ge 0$. Note that

    $$F_Y(y) = P(Y \le y) = P(G(U) \le y) = P[U \le G^{-1}(y)] = F_U[G^{-1}(y)] = u.$$

    So it must be true that $G^{-1}(y) = 1 - e^{-y^2} = u$ so that $G(u) = [-\ln(1-u)]^{1/2}$. Therefore, the random variable $Y = [-\ln(1\text{-}U)]^{1/2}$ has distribution function $F_Y(y)$.

11. Problem 6.69 from the book (p 332)

a. The joint density is $f(y_1, y_2) = \frac{1}{y_1^2 y_2^2}$, $y_1 > 1, y_2 > 1$.

b. We have that $y_1 = u_1 u_2$ and $y_2 = u_2(1 - u_1)$. The Jacobian of transformation is u_2. So,

$$f(u_1, u_2) = \frac{1}{u_1^2 u_2^3 (1-u_1)^2},$$

with limits as specified in the problem.

c. The limits may be simplified to: $1/u_1 < u_2$, $0 < u_1 < 1/2$, or $1/(1-u_1) < u_2$, $1/2 \le u_1 \le 1$.

d. If $0 < u_1 < 1/2$, then $f_{U_1}(u_1) = \int_{1/u_1}^{\infty} \frac{1}{u_1^2 u_2^3 (1-u_1)^2} du_2 = \frac{1}{2(1-u_1)^2}$.

If $1/2 \le u_1 \le 1$, then $f_{U_1}(u_1) = \int_{1/(1-u_1)}^{\infty} \frac{1}{u_1^2 u_2^3 (1-u_1)^2} du_2 = \frac{1}{2u_1^2}$.

e. Not independent since the joint density does not factor. Also note that the support is not rectangular.

Challenge Question:

Let X be a random variable with a pdf of $f_X(x)$ and a cdf of $F_X(x)$ on the support (a, b). Prove that the distribution of $U = F(X)$ is Uniform$(0, 1)$. *Hint:* The cdf of a random variable is always invertible.