Homework 7

Solutions!

1. Let $X_1, ..., X_n$ be i.i.d. $\Gamma(\alpha, \beta)$. Show that $U = \sum_{i=1}^n X_i$ is sufficient for β when α is known.
Solution:

$$\begin{aligned} X_1, ..., X_n \quad & \text{i.i.d.} \quad \Gamma(\alpha, \beta) \\ \Rightarrow L(x_1, ..., x_n|\alpha, \beta) \quad &= \quad \prod_{i=1}^n \frac{1}{\Gamma(\alpha)\beta^\alpha} x_i^{\alpha-1} e^{-\frac{x_i}{\beta}} \\ &= \quad e^{-\frac{\sum_{i=1}^n x_i}{\beta}} (\frac{1}{\Gamma(\alpha)\beta^\alpha})^n \prod_{i=1}^n x_i^{\alpha-1} \end{aligned}$$

So, by the factorization theorem we have that $U = \sum_{i=1}^n X_i$ is a sufficient statistic where $g(u, \theta) = g(\sum_{i=1}^n x_i, \beta) = e^{-\frac{\sum_{i=1}^n x_i}{\beta}} (\frac{1}{\Gamma(\alpha)\beta^\alpha})^n$ and $h(x_1, ..., x_n) = \prod_{i=1}^n x_i^{\alpha-1}$

2. Let $X_1, ..., X_n$ be i.i.d. $NBinomial(r, p)$. Show that $U = \sum_{i=1}^n X_i$ is sufficient for p when r is known.
Solution:

$$\begin{aligned} X_1, ..., X_n \quad & \text{i.i.d.} \quad NBinomial(r, p) \\ \Rightarrow L(x_1, ..., x_n|\alpha, \beta) \quad &= \quad \prod_{i=1}^n \binom{x_i - 1}{r - 1} p^r (1-p)^{x_i - r} \\ &= \quad p^{nr} (1-p)^{(\sum_{i=1}^n x_i) - nr} \prod_{i=1}^n \binom{x_i - 1}{r - 1} \end{aligned}$$

So, by the factorization theorem we have that $U = \sum_{i=1}^n X_i$ is a sufficient statistic where $g(u, \theta) = g(\sum_{i=1}^n x_i, p) = p^{nr}(1-p)^{(\sum_{i=1}^n x_i) - nr}$ and $h(x_1, ..., x_n) = \prod_{i=1}^n \binom{x_i - 1}{r - 1}$

3. Let $X_1, ..., X_n$ be i.i.d. $U(0, \theta)$. Show that $Y_{(n)}$ is sufficient for θ
Solution:

$$
\begin{aligned}
X_1, ..., X_n \quad \text{i.i.d.} \quad & U(0,\theta) \\
\Rightarrow L(x_1, ..., x_n|\alpha,\beta) &= \prod_{i=1}^{n} \frac{1}{\theta} I(0 < x_i < \theta) \\
&= \frac{1}{\theta^n} \prod_{i=1}^{n} I(0 < x_i < \theta) \\
&= \frac{1}{\theta^n} I(0 < x_1, x_2, ..., x_n < \theta) \\
&= \frac{1}{\theta^n} I(0 < x_{(1)} < x_{(n)} < \theta) \\
&= \frac{1}{\theta^n} I(x_{(n)} < \theta) I(0 < x_{(1)})
\end{aligned}
$$

So, by the factorization theorem we have that $U = X_{(n)}$ is a sufficient statistic where $g(u,\theta) = g(x_{(n)},\theta) = \frac{1}{\theta^n} I(x_{(n)} < \theta)$ and $h(x_1, ..., x_n) = I(0 < x_{(1)})$

4. Let $X_1, ..., X_n$ be i.i.d. $Exp(a\delta + b)$.

   a) Show that $U = \sum_{i=1}^{n} X_i$ is a sufficient statistic for δ when $a > 0$ and $b > 0$, and are both known values.
   Solution:

$$
\begin{aligned}
L(x_1, ..., x_n|\delta) &= \prod_{i=1}^{n} \frac{1}{(a\delta + b)} e^{-\frac{x_i}{a\delta+b}} I(0 < x_i) \\
&= \frac{1}{(a\delta + b)^n} e^{-\frac{\sum_{i=1}^{n} x_i}{a\delta+b}} I(0 < x_{(1)})
\end{aligned}
$$

   Here we have $g(\sum_{i=1}^{n} x_i, \delta) = \frac{1}{(a\delta+b)^n} e^{-\frac{\sum_{i=1}^{n} x_i}{a\delta+b}}$ and $h(x_1, ..., x_n) = I(0 < x_{(1)})$. Thus, by the factorization theorem we see that $U = \sum_{i=1}^{n} X_i$ is sufficient for δ.

   b) An unbiased estimator for δ in this problem is $(\overline{X} - b)/a$ where $\overline{X} = \left(\sum_{i=1}^{n} X_i\right)/n$. Can this unbiased estimator be improved using the Rao-Blackwell theorem? Explain your answer.
   Solution:
   The Rao-Blackwell theorem states that given an unbiased estimator $\hat{\delta}$ for δ and a sufficient statistic U, then $\hat{\delta}^* = E[\hat{\delta}|U = u]$ will have the property $V[\hat{\delta}^*] \leq V[\hat{\delta}]$, but we can see in the proof of this property we have $V[\hat{\delta}] = V[\hat{\delta}^*] + E[V[\hat{\delta}|U = u]]$. Since $\hat{\delta} = (\overline{X} - b)/a$ is a function of $U = \sum_{i=1}^{n} X_i$, we see that $\hat{\delta}|U = u$ is a constant ($[u/n - b]/a$), and thus $V[\hat{\delta}|U = u] = 0$, leaving $V[\hat{\delta}] = V[\hat{\delta}^*]$. Therefore, the Rao-Blackwell theorem cannot be applied to improve the estimator.

5. Problem 9.38 from the book (p 462)

Solution:
For this exercise, the likelihood function is given by

$$L = \frac{1}{(2\pi)^{n/2}\sigma^n} \exp\left[-\frac{\sum_{i=1}^n (y_i - \mu)^2}{2\sigma^2}\right] = (2\pi)^{-n/2}\sigma^{-n}\exp\left[\frac{-1}{2\sigma^2}\left(\sum_{i=1}^n y_i^2 - 2\mu n\bar{y} + n\mu^2\right)\right].$$

a. When σ^2 is known, $\bar{Y}$ is sufficient for μ by Theorem 9.4 with
$g(\bar{y},\mu) = \exp\left(\frac{2\mu n\bar{y} - n\mu^2}{2\sigma^2}\right)$ and $h(\mathbf{y}) = (2\pi)^{-n/2}\sigma^{-n}\exp\left(\frac{-1}{2\sigma^2}\sum_{i=1}^n y_i^2\right)$.

b. When μ is known, use Theorem 9.4 with
$g(\sum_{i=1}^n (y_i - \mu)^2, \sigma^2) = (\sigma^2)^{-n/2}\exp\left[-\frac{\sum_{i=1}^n (y_i - \mu)^2}{2\sigma^2}\right]$ and $h(\mathbf{y}) = (2\pi)^{-n/2}$.

c. When both μ and σ^2 are unknown, the likelihood can be written in terms of the two statistics $U_1 = \sum_{i=1}^n Y_i$ and $U_2 = \sum_{i=1}^n Y_i^2$ with $h(\mathbf{y}) = (2\pi)^{-n/2}$. The statistics $\bar{Y}$ and S^2 are also jointly sufficient since they can be written in terms of U_1 and U_2.

6. Problem 9.40 from the book (p 463)

Solution:

The likelihood is $L(\theta) = 2^n\theta^{-n}\prod_{i=1}^n y_i \exp\left(-\sum_{i=1}^n y_i^2/\theta\right)$. By Theorem 9.4, $U = \sum_{i=1}^n Y_i^2$ is sufficient for θ with $g(u,\theta) = \theta^{-n}\exp(-u/\theta)$ and $h(\mathbf{y}) = 2^n\prod_{i=1}^n y_i$.

7. Problem 9.41 from the book (p 463)

Solution:

The likelihood is $L(\alpha) = \alpha^{-n}m^n\left(\prod_{i=1}^n y_i\right)^{m-1}\exp\left(-\sum_{i=1}^n y_i^m/\alpha\right)$. By Theorem 9.4, $U = \sum_{i=1}^n Y_i^m$ is sufficient for α with $g(u,\alpha) = \alpha^{-n}\exp(-u/\alpha)$ and $h(\mathbf{y}) = m^n\left(\prod_{i=1}^n y_i\right)^{m-1}$.

8. Problem 9.42 from the book (p 463)

Solution:

The likelihood function is $L(p) = p^n(1-p)^{\Sigma y_i - n} = p^n(1-p)^{n\bar{y}-n}$. By Theorem 9.4, $\bar{Y}$ is sufficient for p with $g(\bar{y}, p) = p^n(1-p)^{n\bar{y}-n}$ and $h(\mathbf{y}) = 1$.

9. Problem 9.43 from the book (p 463)

Solution:

With θ known, the likelihood is $L(\alpha) = \alpha^n\theta^{-n\alpha}\left(\prod_{i=1}^n y_i\right)^{\alpha-1}$. By Theorem 9.4, $U = \prod_{i=1}^n Y_i$ is sufficient for α with $g(u,\alpha) = \alpha^n\theta^{-n\alpha}\left(\prod_{i=1}^n y_i\right)^{\alpha-1}$ and $h(\mathbf{y}) = 1$.

10. Problem 9.44 from the book (p 463)

    Solution:

    With β known, the likelihood is $L(\alpha) = \alpha^n \beta^{n\alpha} \left(\prod_{i=1}^n y_i \right)^{-(\alpha+1)}$. By Theorem 9.4, $U = \prod_{i=1}^n Y_i$ is sufficient for α with $g(u,\alpha) = \alpha^n \beta^{n\alpha} (u)^{-(\alpha+1)}$ and $h(\mathbf{y}) = 1$.

11. Problem 9.45 from the book (p 463)

    Solution:

    The likelihood function is
    $$L(\theta) = \prod_{i=1}^n f(y_i \mid \theta) = [a(\theta)]^n \left[\prod_{i=1}^n b(y_i) \right] \exp\left[-c(\theta) \sum_{i=1}^n d(y_i) \right].$$
    Thus, $U = \sum_{i=1}^n d(Y_i)$ is sufficient for θ because by Theorem 9.4 $L(\theta)$ can be factored into, where $u = \sum_{i=1}^n d(y_i)$, $g(u,\theta) = [a(\theta)]^n \exp[-c(\theta)u]$ and $h(\mathbf{y}) = \prod_{i=1}^n b(y_i)$.

12. Problem 9.46 from the book (p 463)

    Solution:

    The exponential distribution is in exponential form since $a(\beta) = c(\beta) = 1/\beta$, $b(y) = 1$, and $d(y) = y$. Thus, by Ex. 9.45, $\sum_{i=1}^n Y_i$ is sufficient for β, and then so is $\overline{Y}$.

Challenge Question:

Let $X_1, ..., X_n$ be i.i.d. $U(\theta_1, \theta_2)$. Show that $\langle X_{(1)}, X_{(n)} \rangle$ together are jointly sufficient for $\langle \theta_1, \theta_2 \rangle$. *Hint:* To show two statistics are jointly sufficient you can use the factorization theorem by showing that the likelihood can be written as the product of a function of just the data alone and a function of just the proposted staistics and the unknown parameters.