Homework 4

Solutions

Book problems:

1. Problem 8.1 from the book (p 394)

   Solution:

   Let $B = B(\hat{\theta})$. Then,

   $$MSE(\hat{\theta}) = E\left[(\hat{\theta}-\theta)^2\right] = E\left[(\hat{\theta}-E(\hat{\theta})+B)^2\right] = E\left[\left(\hat{\theta}-E(\hat{\theta})\right)^2\right] + E(B^2) + 2B\times E\left[\hat{\theta}-E(\hat{\theta})\right]$$
   $$= V(\hat{\theta}) + B^2.$$

2. Problem 8.2 from the book (p 394)

   Solution:

   a. The estimator $\hat{\theta}$ is unbiased if $E(\hat{\theta}) = \theta$. Thus, $B(\hat{\theta}) = 0$.

   b. $E(\hat{\theta}) = \theta + 5$.

3. Problem 8.3 from the book (p 394)

   Solution:

   a. Using Definition 8.3, $B(\hat{\theta}) = a\theta + b - \theta = (a-1)\theta + b$.

   b. Let $\hat{\theta}^* = (\hat{\theta} - b)/a$.

4. Problem 8.4 from the book (p 394)

   Solution:

   a. They are equal.

   b. $MSE(\hat{\theta}) > V(\hat{\theta})$.

5. Problem 8.5 from the book (p 394)

   Solution:

a. Note that $E(\hat{\theta}^*) = \theta$ and $V(\hat{\theta}^*) = V[(\hat{\theta} - b)/a] = V(\hat{\theta})/a^2$. Then,

$$\text{MSE}(\hat{\theta}^*) = V(\hat{\theta}^*) = V(\hat{\theta})/a^2 .$$

b. Note that $\text{MSE}(\hat{\theta}) = V(\hat{\theta}) + B(\hat{\theta}) = V(\hat{\theta}) + [(a-1)\theta + b]^2$. A sufficiently large value of a will force $\text{MSE}(\hat{\theta}^*) < \text{MSE}(\hat{\theta})$. Example: $a = 10$.

c. A amply small value of a will make $\text{MSE}(\hat{\theta}^*) > \text{MSE}(\hat{\theta})$. Example: $a = .5$, $b = 0$.

6. Problem 8.6 from the book (p 394)

Solution:

a. $E(\hat{\theta}_3) = aE(\hat{\theta}_1) + (1-a)E(\hat{\theta}_2) = a\theta + (1-a)\theta = \theta$.

b. $V(\hat{\theta}_3) = a^2 V(\hat{\theta}_1) + (1-a)^2 V(\hat{\theta}_2) = a^2\sigma_1^2 + (1-a)\sigma_2^2$, since it was assumed that $\hat{\theta}_1$ and $\hat{\theta}_2$ are independent. To minimize $V(\hat{\theta}_3)$, we can take the first derivative (with respect to a), set it equal to zero, to find

$$a = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} .$$

(One should verify that the second derivative test shows that this is indeed a minimum.)

7. Problem 8.7 from the book (p 394)

Solution:

Following Ex. 8.6 but with the condition that $\hat{\theta}_1$ and $\hat{\theta}_2$ are not independent, we find

$$V(\hat{\theta}_3) = a^2\sigma_1^2 + (1-a)\sigma_2^2 + 2a(1-a)c .$$

Using the same method w/ derivatives, the minimum is found to be

$$a = \frac{\sigma_2^2 - c}{\sigma_1^2 + \sigma_2^2 - 2c} .$$

8. Problem 8.8 from the book (p 394)

Solution:

a. Note that $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}_3$ and $\hat{\theta}_5$ are simple linear combinations of Y_1, Y_2, and Y_3. So, it is easily shown that all four of these estimators are *unbiased*. From Ex. 6.81 it was shown that $\hat{\theta}_4$ has an exponential distribution with mean $\theta/3$, so this estimator is biased.

b. It is easily shown that $V(\hat{\theta}_1) = \theta^2$, $V(\hat{\theta}_2) = \theta^2/2$, $V(\hat{\theta}_3) = 5\theta^2/9$, and $V(\hat{\theta}_5) = \theta^2/9$, so the estimator $\hat{\theta}_5$ is unbiased and has the smallest variance.

9. Problem 8.20 from the book (p 396)

Solution:

If Y has an exponential distribution with mean θ, then by Ex. 4.11, $E(\sqrt{Y}) = \sqrt{\pi\theta}/2$.

a. Since Y_1 and Y_2 are independent, $E(X) = \pi\theta/4$ so that $(4/\pi)X$ is unbiased for θ.

b. Following part a, it is easily seen that $E(W) = \pi^2\theta^2/16$, so $(4^2/\pi^2)W$ is unbiased for θ^2.

10. Let $(X_1, X_2) \sim N_2(0, 0, 1, 1, \rho)$ (i.e. X_1 and X_2 share a Bi-Variate normal distribution)

a) Show that $\hat{\rho} = X_1X_2$ is an unbiased estimator for ρ (pronounced Rho) *Hint:* The conditional Expectation Theorem can be used here

Solution:

First, note that since $(X_1, X_2) \sim N_2(0, 0, 1, 1, \rho)$, we know that (from crib-sheet/last semeseter)

$$\begin{aligned} X_1 &\sim N(0,1) \text{ and} \\ X_2|X_1 = x_1 &\sim N(\rho(x_1), 1-\rho^2). \end{aligned}$$

Each implies $E[X_1] = 0$ and $E[X_2|X_1 = x_1] = \rho x_1$, respectively. Thus we can see

$$\begin{aligned} E[X_1X_2] &= E[E[X_1X_2|X_1 = x_1]] \\ &= E[X_1E[X_2|X_1 = x_1]] \\ &= E[X_1\rho X_1] \\ &= \rho E[X_1^2] \\ &= \rho(V[X_1] + E^2[X_1]) \\ &= \rho \end{aligned}$$

and thus X_1X_2 is an unbiased estimator for ρ.

b) Find $MSE(\hat{\rho})$. *Hint:* The conditional Variance Theorem can be used here

Solution:

Using the marginal and conditional distributions discussed in the previous part, we see that

$$\begin{aligned} MSE(\hat{\rho}) &= MSE(X_1X_2) \\ &= V[X_1X_2] + B^2(X_1X_2) \\ &= V[X_1X_2] \\ &= E[V[X_1X_2|X_1 = x_1]] + V[E[X_1X_2|X_1 = x_1]] \end{aligned}$$

$$
\begin{aligned}
&= E[X_1^2 V[X_2|X_1 = x_1]] + V[X_1 E[X_2|X_1 = x_1]] \\
&= E[X_1^2(1-\rho^2)] + V[X_1 X_1 \rho] \leftarrow \text{Since } X_2|X_1 = x_1 \sim N(\rho x_1, 1-\rho^2) \\
&= (1-\rho^2)E[X_1^2] + \rho^2 V[X_1^2] \\
&= 1 - \rho^2 + 2\rho^2 \leftarrow \text{Since } X_1 \sim N(0,1) \Rightarrow X_1^2 \sim \chi_1^2 \\
&= 1 + \rho^2
\end{aligned}
$$

11. Let $X_1, ...X_n$ be i.i.d. $Exp(\delta)$ where $\delta > 0$ is unknown.

a) Show that $\bar{X} = \frac{\sum_{i=1}^n X_i}{n}$ is an unbiased estimator for δ

Solution:

Proof.

$$
\begin{aligned}
E[\bar{X}] &= E[\frac{\sum_{i=1}^n X_i}{n}] \\
&= \frac{\sum_{i=1}^n E[X_i]}{n} \\
&= \frac{\sum_{i=1}^n \delta}{n} \leftarrow \text{Since } X_1, ..., X_n \text{ are i.i.d. } Exp(\delta) \\
&= \frac{n\delta}{n} \\
&= \delta \\
\Rightarrow B(\bar{X}) &= \delta - \delta \\
&= 0
\end{aligned}
$$

Thus $\bar{X}$ is an unbiased estimator for δ. □

b) Show that $nX_{(1)} = n \times \min(X_1, ..., X_n)$ is an unbiased estimator for δ

Solution:

Proof. First, either claim $X_{(1)} \sim Exp(\delta/n)$ from class, or prove that $X_{(1)} \sim Exp(\delta/n)$ as below:

$$
\begin{aligned}
f_{X_{(1)}}(x) &= n(1 - F(x))^{n-1} f(x) \\
&= \begin{cases} n(1-(1-e^{-x/\delta}))^{n-1} e^{-x/\delta} & 0 < x < \infty \\ 0 & else \end{cases} \leftarrow \text{Since } X_1, ..., X_n \text{ are iid } Exp(\delta) \\
&= \begin{cases} ne^{-nx/\delta} & 0 < x < \infty \\ 0 & else \end{cases}
\end{aligned}
$$

Since $X_{(1)}$ has the PDF of $Exp(\delta/n)$, we conclude that $X_{(1)} \sim Exp(\delta/n)$. Thus

$$\begin{aligned}
E[nX_{(1)}] &= nE[X_{(1)}] \\
&= n(\delta/n) \leftarrow \text{Since } X_{(1)} \sim Exp(\delta/n) \\
&= \delta \\
\Rightarrow B(nX_{(1)}) &= \delta - \delta \\
&= 0
\end{aligned}$$

Thus $nX_{(1)}$ is an unbiased estimator for δ. □

c) Which estimator is better for estimating δ, $\bar{X}$ or $nX_{(1)}$?
Solution:

Proof. Note: Since $X_1, ..., X_n$ are i.i.d. $Exp(\delta)$, we see that $\sum_{i=1}^n X_i \sim \Gamma(n, \delta)$

$$\begin{aligned}
MSE(\bar{X}) &= V[\bar{X}] + B^2(\bar{X}) \\
&= V[\bar{X}] \\
&\quad \uparrow \text{Since } \bar{X} \text{ is unbiased} \\
&= \frac{1}{n^2} V[\sum_{i=1}^n X_i] \\
&= \frac{1}{n^2} n\delta^2 \\
&\quad \uparrow \text{Since } \textstyle\sum_{i=1}^n X_i \sim \Gamma(n, \delta) \\
&= \frac{\delta^2}{n}
\end{aligned}$$

$$\begin{aligned}
MSE(nX_{(1)}) &= V[nX_{(1)}] + B^2(nX_{(1)}) \\
&= V[nX_{(1)}] \leftarrow \text{Since} \\
&\qquad nX_{(1)} \text{ is unbiased} \\
&= n^2 V[X_{(1)}] \\
&= n^2 \frac{\delta^2}{n^2} \\
&\quad \uparrow \text{Since } X_{(1)} \sim Exp(\delta/n) \\
&= \delta^2
\end{aligned}$$

Since both estimators are unbiased and $MSE(\bar{X}) < MSE(nX_{(1)})$ we conclude that $\bar{X}$ is the better estimator. □

Challenge Question:
Let $X_1, ...X_n$ be i.i.d. $U(-\theta, \theta)$. Find an unbiased estimator $\hat{\theta}$ that is a function of $X_1, ..., X_n$