Homework 9

SOLUTIONS!

Let $X_1, ..., X_n$ be i.i.d. $N(a\mu + b, \sigma^2 + c)$

1. Assume that σ^2 is known. Construct a two sided CI for μ based on $X_1, ..., X_n$ when:

   a) $a = 1$, $b > 0$, $c = 0$

   Solution:

   (Let $Z_{\alpha/2}$ be the $1 - \alpha/2$ percentile of the $N(0, 1)$ distribution).

$$
\begin{aligned}
\bar{X} &\sim N(\mu + b, \frac{\sigma^2}{n}) \\
\Rightarrow \sqrt{n}\frac{\bar{X} - (\mu + b)}{\sigma} &\sim N(0, 1) \\
\Rightarrow P(-Z_{\alpha/2} < \sqrt{n}\frac{\bar{X} - (\mu + b)}{\sigma} < Z_{\alpha/2}) &= 1 - \alpha \\
\Rightarrow P(\bar{X} - \frac{\sigma}{\sqrt{n}} Z_{\alpha/2} < (\mu + b) < \bar{X} + \frac{\sigma}{\sqrt{n}} Z_{\alpha/2}) &= 1 - \alpha \\
\Rightarrow P(\bar{X} - \frac{\sigma}{\sqrt{n}} Z_{\alpha/2} - b < \mu < \bar{X} + \frac{\sigma}{\sqrt{n}} Z_{\alpha/2} - b) &= 1 - \alpha
\end{aligned}
$$

   So, our two sided $1 - \alpha$ CI for μ is $((\bar{X} - b) - \frac{\sigma}{\sqrt{n}} Z_{\alpha/2}, (\bar{X} - b) + \frac{\sigma}{\sqrt{n}} Z_{\alpha/2})$

   b) $a > 0$, $b = 0$, $c = 0$

   Solution:

(Let $Z_{\alpha/2}$ be the $1-\alpha/2$ percentile of the $N(0,1)$ distribution).

$$\begin{aligned}
\bar{X} &\sim N(a\mu, \frac{\sigma^2}{n}) \\
\Rightarrow \sqrt{n}\frac{\bar{X}-(a\mu)}{\sigma} &\sim N(0,1) \\
\Rightarrow P(-Z_{\alpha/2} < \sqrt{n}\frac{\bar{X}-(a\mu)}{\sigma} < Z_{\alpha/2}) &= 1-\alpha \\
\Rightarrow P(\bar{X} - \frac{\sigma}{\sqrt{n}}Z_{\alpha/2} < (a\mu) < \bar{X} + \frac{\sigma}{\sqrt{n}}Z_{\alpha/2}) &= 1-\alpha \\
\Rightarrow P(\frac{1}{a}(\bar{X} - \frac{\sigma}{\sqrt{n}}Z_{\alpha/2}) < \mu < \frac{1}{a}(\bar{X} + \frac{\sigma}{\sqrt{n}}Z_{\alpha/2})) &= 1-\alpha
\end{aligned}$$

So, our two sided $1-\alpha$ CI for μ is $(\frac{1}{a}(\bar{X} - \frac{\sigma}{\sqrt{n}}Z_{\alpha/2}), \frac{1}{a}(\bar{X} + \frac{\sigma}{\sqrt{n}}Z_{\alpha/2}))$

c) $a = 1$, $b = 0$, $c > 0$
Solution:
(Let $Z_{\alpha/2}$ be the $1-\alpha/2$ percentile of the $N(0,1)$ distribution).

$$\begin{aligned}
\bar{X} &\sim N(\mu, \frac{\sigma^2+c}{n}) \\
\Rightarrow \sqrt{n}\frac{\bar{X}-\mu}{\sqrt{\sigma^2+c}} &\sim N(0,1) \\
\Rightarrow P(-Z_{\alpha/2} < \sqrt{n}\frac{\bar{X}-\mu}{\sqrt{\sigma^2+c}} < Z_{\alpha/2}) &= 1-\alpha \\
\Rightarrow P(\bar{X} - \frac{\sqrt{\sigma^2+c}}{\sqrt{n}}Z_{\alpha/2} < \mu < \bar{X} + \frac{\sqrt{\sigma^2+c}}{\sqrt{n}}Z_{\alpha/2}) &= 1-\alpha
\end{aligned}$$

So, our two sided $1-\alpha$ CI for μ is $(\bar{X} - \frac{\sqrt{\sigma^2+c}}{\sqrt{n}}Z_{\alpha/2}, \bar{X} + \frac{\sqrt{\sigma^2+c}}{\sqrt{n}}Z_{\alpha/2})$

2. Assume that σ^2 is unkown. Contruct a two sided CI for μ based on $X_1, ..., X_n$ when:

   a) $a > 0$, $b > 0$, $c = 0$
   Solution:

(Let $t_{\alpha/2}$ be the $1-\alpha/2$ percentile of the t_{n-1} distribution).

$$\begin{aligned}
\bar{X} &\sim N(a\mu+b, \frac{\sigma^2}{n}) \\
\Rightarrow \sqrt{n}\frac{\bar{X}-(a\mu+b)}{\sigma} &\sim N(0,1) \\
\to \frac{n-1}{\sigma^2}S_X^2 &\sim \chi^2_{n-1} \\
\Rightarrow \frac{\sqrt{n}\frac{\bar{X}-(a\mu+b)}{\sigma}}{\sqrt{\frac{n-1}{\sigma^2}S_X^2/(n-1)}} &= \frac{\sqrt{n}(\bar{X}-(a\mu+b))}{\sqrt{S_X^2}} \\
&\sim t_{n-1} \\
\Rightarrow P(-t_{\alpha/2} < \frac{\sqrt{n}(\bar{X}-(a\mu+b))}{\sqrt{S_X^2}} < t_{\alpha/2}) &= 1-\alpha \\
\Rightarrow P(\bar{X} - \frac{\sqrt{S_X^2}}{\sqrt{n}}t_{\alpha/2} < a\mu+b < \bar{X} + \frac{\sqrt{S_X^2}}{\sqrt{n}}t_{\alpha/2}) &= 1-\alpha \\
\Rightarrow P(\frac{1}{a}(\bar{X} - \frac{\sqrt{S_X^2}}{\sqrt{n}}t_{\alpha/2} - b) < \mu < \frac{1}{a}(\bar{X} + \frac{\sqrt{S_X^2}}{\sqrt{n}}t_{\alpha/2} - b)) &= 1-\alpha
\end{aligned}$$

So, our two sided $1-\alpha$ CI for μ is $(\frac{1}{a}((\bar{X}-b)-\frac{\sqrt{S_X^2}}{\sqrt{n}}t_{\alpha/2}), \frac{1}{a}((\bar{X}-b)+\frac{\sqrt{S_X^2}}{\sqrt{n}}t_{\alpha/2}))$

b) $a>0$, $b>0$, $c>0$
Solution:

(Let $t_{\alpha/2}$ be the $1-\alpha/2$ percentile of the t_{n-1} distribution).

$$\begin{aligned}
\bar{X} &\sim N(a\mu+b, \frac{\sigma^2+c}{n}) \\
\Rightarrow \sqrt{n}\frac{\bar{X}-(a\mu+b)}{\sqrt{\sigma^2+c}} &\sim N(0,1) \\
\rightarrow \frac{n-1}{\sigma^2+c}S_X^2 &\sim \chi^2_{n-1} \\
\Rightarrow \frac{\sqrt{n}\frac{\bar{X}-(a\mu+b)}{\sqrt{\sigma^2+c}}}{\sqrt{\frac{n-1}{\sigma^2+c}S_X^2/(n-1)}} &= \frac{\frac{1}{\sqrt{\sigma^2+c}}}{\frac{1}{\sqrt{\sigma^2+c}}}\frac{\sqrt{n}(\bar{X}-(a\mu+b))}{\sqrt{S_X^2}} \\
&= \frac{\sqrt{n}(\bar{X}-(a\mu+b))}{\sqrt{S_X^2}} \\
&\sim t_{n-1} \\
\Rightarrow P(-t_{\alpha/2} < \frac{\sqrt{n}(\bar{X}-(a\mu+b))}{\sqrt{S_X^2}} < t_{\alpha/2}) &= 1-\alpha \\
\Rightarrow P(\bar{X}-\frac{\sqrt{S_X^2}}{\sqrt{n}}t_{\alpha/2} < a\mu+b < \bar{X}+\frac{\sqrt{S_X^2}}{\sqrt{n}}t_{\alpha/2}) &= 1-\alpha \\
\Rightarrow P(\frac{1}{a}(\bar{X}-\frac{\sqrt{S_X^2}}{\sqrt{n}}t_{\alpha/2}-b) < \mu < \frac{1}{a}(\bar{X}+\frac{\sqrt{S_X^2}}{\sqrt{n}}t_{\alpha/2}-b)) &= 1-\alpha
\end{aligned}$$

So, our two sided $1-\alpha$ CI for μ is $(\frac{1}{a}((\bar{X}-b)-\frac{\sqrt{S_X^2}}{\sqrt{n}}t_{\alpha/2}), \frac{1}{a}((\bar{X}-b)+\frac{\sqrt{S_X^2}}{\sqrt{n}}t_{\alpha/2}))$

3. Assume that σ^2 is unkown, $a=1$, and $b=0$.

   a) Construct a two sided CI based on $X_1, ..., X_n$ for σ^2 when $c=0$
   Solution:
   (Let $\chi^2_{\alpha/2}$ and $\chi^2_{1-\alpha/2}$ be the $1-\alpha/2$ and $\alpha/2$ percentiles of the χ^2_{n-1} distribution).

$$\begin{aligned}
X_1, ..., X_n &\quad \text{i.i.d.} \quad N(\mu, \sigma^2) \\
\Rightarrow \frac{n-1}{\sigma^2}S_X^2 &\sim \chi^2_{n-1} \\
\Rightarrow P(\chi^2_{1-\alpha/2} < \frac{n-1}{\sigma^2}S_X^2 < \chi^2_{\alpha/2}) &= 1-\alpha \\
\Rightarrow P(\frac{1}{(n-1)S_X^2}\chi^2_{1-\alpha/2} < \frac{1}{\sigma^2} < \frac{1}{(n-1)S_X^2}\chi^2_{\alpha/2}) &= 1-\alpha \\
\Rightarrow P(\frac{(n-1)S_X^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)S_X^2}{\chi^2_{1-\alpha/2}}) &= 1-\alpha
\end{aligned}$$

   So, our two sided $1-\alpha$ CI for σ^2 is $(\frac{(n-1)S_X^2}{\chi^2_{\alpha/2}}, \frac{(n-1)S_X^2}{\chi^2_{1-\alpha/2}})$

b) Construct a two sided CI based on $X_1, ..., X_n$for σ^2 when $c = 1$

Solution:

(Let χ^2_α and $\chi^2_{1-\alpha}$ be the $1-\alpha$ and α percentiles of the χ^2_{n-1} distribution).

$$\begin{aligned}
X_1, ..., X_n &\quad \text{i.i.d.} \quad N(\mu, \sigma^2 + 1) \\
\Rightarrow \frac{n-1}{\sigma^2+1} S_X^2 &\quad \sim \quad \chi^2_{n-1} \\
\Rightarrow P(\chi^2_{1-\alpha/2} < \frac{n-1}{\sigma^2+1} S_X^2 < \chi^2_{\alpha/2}) &\quad = \quad 1-\alpha \\
\Rightarrow P(\frac{1}{(n-1)S_X^2}\chi^2_{1-\alpha/2} < \frac{1}{\sigma^2+1} < \frac{1}{(n-1)S_X^2}\chi^2_{\alpha/2}) &\quad = \quad 1-\alpha \\
\Rightarrow P(\frac{(n-1)S_X^2}{\chi^2_{\alpha/2}} < \sigma^2 + 1 < \frac{(n-1)S_X^2}{\chi^2_{1-\alpha/2}}) &\quad = \quad 1-\alpha \\
\Rightarrow P(\frac{(n-1)S_X^2}{\chi^2_{\alpha/2}} - 1 < \sigma^2 < \frac{(n-1)S_X^2}{\chi^2_{1-\alpha/2}} - 1) &\quad = \quad 1-\alpha
\end{aligned}$$

So, our two sided $1-\alpha$ CI for σ^2 is $(\frac{(n-1)S_X^2}{\chi^2_{\alpha/2}} - 1, \frac{(n-1)S_X^2}{\chi^2_{1-\alpha/2}} - 1)$

c) Construct a lower bound CI based on $X_1, ..., X_n$ for σ^2 when $c = 0$

Solution:

(Let χ^2_α and $\chi^2_{1-\alpha}$ be the $1-\alpha$ and α percentiles of the χ^2_{n-1} distribution).

$$\begin{aligned}
X_1, ..., X_n &\quad \text{i.i.d.} \quad N(\mu, \sigma^2) \\
\Rightarrow \frac{n-1}{\sigma^2} S_X^2 &\quad \sim \quad \chi^2_{n-1} \\
\Rightarrow P(\frac{n-1}{\sigma^2} S_X^2 < \chi^2_\alpha) &\quad = \quad 1-\alpha \\
\Rightarrow P(\frac{1}{\sigma^2} < \frac{1}{(n-1)S_X^2}\chi^2_\alpha) &\quad = \quad 1-\alpha \\
\Rightarrow P(\frac{(n-1)S_X^2}{\chi^2_\alpha} < \sigma^2) &\quad = \quad 1-\alpha
\end{aligned}$$

So, our lower bound $1-\alpha$ CI for σ^2 is $(\frac{(n-1)S_X^2}{\chi^2_\alpha}, \infty)$

d) Construct an upper bound CI based on $X_1, ..., X_n$ for σ^2 when $c = 1$

Solution:

(Let χ^2_α and $\chi^2_{1-\alpha}$ be the $1-\alpha$ and α percentiles of the χ^2_{n-1} distribution).

$$
\begin{aligned}
X_1, ..., X_n &\quad \text{i.i.d.} \quad N(\mu, \sigma^2+1) \\
\Rightarrow \frac{n-1}{\sigma^2+1} S_X^2 &\quad \sim \quad \chi^2_{n-1} \\
\Rightarrow P(\chi^2_{1-\alpha} < \frac{n-1}{\sigma^2+1} S_X^2) &\quad = \quad 1-\alpha \\
\Rightarrow P(\frac{1}{(n-1)S_X^2}\chi^2_{1-\alpha} < \frac{1}{\sigma^2+1}) &\quad = \quad 1-\alpha \\
\Rightarrow P(\sigma^2+1 < \frac{(n-1)S_X^2}{\chi^2_{1-\alpha}}) &\quad = \quad 1-\alpha \\
\Rightarrow P(\sigma^2 < \frac{(n-1)S_X^2}{\chi^2_{1-\alpha}-1}) &\quad = \quad 1-\alpha
\end{aligned}
$$

So, our upper bound $1-\alpha$ CI for σ^2 is $(-\infty, \frac{(n-1)S_X^2}{\chi^2_{1-\alpha}} - 1)$ (which we can write as $(0, \frac{(n-1)S_X^2}{\chi^2_{1-\alpha}} - 1)$ since $\sigma^2 > 0$)

4. Problem 8.80 from the book (p 430)

   Solution:

   The 95% CI, based on a t–distribution with 21 – 1 = 20 degrees of freedom, is
   $$26.6 \pm 2.086\left(7.4/\sqrt{21}\right) = 26.6 \pm 3.37 \text{ or } (23.23, 29.97).$$

5. Problem 8.81 from the book (p 430)

   Solution:

   The sample statistics are $\bar{y} = 60.8$, $s = 7.97$. So, the 95% CI is
   $$60.8 \pm 2.262\left(7.97/\sqrt{10}\right) = 60.8 \pm 5.70 \text{ or } (55.1, 66.5).$$

6. Problem 8.93 from the book (p 433)

   Solution:

a. Since the two random samples are assumed to be independent and normally distributed, the quantity $2\overline{X} + \overline{Y}$ is normally distributed with mean $2\mu_1 + \mu_2$ and variance $\left(\frac{4}{n} + \frac{3}{m}\right)\sigma^2$. Thus, is σ^2 is known, then $2\overline{X} + \overline{Y} \pm 1.96\sigma\sqrt{\frac{4}{n} + \frac{3}{m}}$ is a 95% CI for $2\mu_1 + \mu_2$.

b. Recall that $(1/\sigma^2)\sum_{i=1}^{n}(X_i - \overline{X})^2$ has a chi–square distribution with $n - 1$ degrees of freedom. Thus, $[1/(3\sigma^2)]\sum_{i=1}^{m}(Y_i - \overline{Y})^2$ is chi–square with $m - 1$ degrees of freedom and the sum of these is chi–square with $n + m - 2$ degrees of freedom. Then, by using Definition 7.2, the quantity

$$T = \frac{2\overline{X} + \overline{Y} - (2\mu_1 + \mu_2)}{\hat{\sigma}\sqrt{\frac{4}{n} + \frac{3}{m}}}, \text{ where}$$

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(X_i - \overline{X})^2 + \frac{1}{3}\sum_{i=1}^{m}(Y_i - \overline{Y})^2}{n + m - 2}.$$

Then, the 95% CI is given by $2\overline{X} + \overline{Y} \pm t_{.025}\hat{\sigma}\sqrt{\frac{4}{n} + \frac{3}{m}}$.

Challenge Question:

Let $X_1, ...X_{n_1}$ be i.i.d. with the following distirbution function:

$$f(x) = \frac{1}{\delta}e^{-\frac{(x-\mu)}{\delta}} I(\mu < x < \infty)$$

Assuming that δ is known, construct a two sided $1 - \alpha$ CI for μ that is a function of $X_1, ..., X_n$, δ, and percentiles of the $Exp(\frac{1}{n})$ distribution.