Homework 5

SOLUTIONS!

1. Let $Y_1, \ldots, Y_n$ denote a random sample from the distribution with the PDF

$$f_X(x) = \begin{cases} (\theta+1)y^\theta & 0 < y < 1 \\ 0 & else \end{cases}$$

Where $\theta > -1$. Find the method of moments estimator for θ

Solution:

First, we see that the distribution shared by $Y_1, \ldots, Y_n$ is $Beta(\theta + 1, 1)$ Which means that the first moment is $\frac{\theta+1}{\theta+2}$. So, we set the first moment equal to the first smaple moment ($\overline{Y} = \frac{\sum_{i=1}^n Y_i}{n}$) to get our estimator

$$\begin{aligned} \overline{Y} &= \frac{\theta+1}{\theta+2} \\ \Rightarrow \overline{Y}\theta + 2\overline{Y} &= \theta + 1 \\ \Rightarrow \theta(\overline{Y} - 1) &= 1 - 2\overline{Y} \\ \Rightarrow \theta &= \frac{1 - 2\overline{Y}}{\overline{Y} - 1} \end{aligned}$$

2. Problem 9.71 from the book (p 475)

Solution:

Since $E(Y) = \mu_1' = 0$ and $E(Y^2) = \mu_2' = V(Y) = \sigma^2$, we have that $\hat{\sigma}^2 = m_2' = \frac{1}{n}\sum_{i=1}^n Y_i^2$.

3. Problem 9.74 part a from the book (p 475)

Solution:

a. First, calculate $\mu_1' = E(Y) = \int_0^\theta 2y(\theta - y)/\theta^2 dy = \theta/3$. Thus, the MOM estimator of θ is

$\hat{\theta} = 3\overline{Y}$.

4. Problem 9.79 from the book (p 476)

Solution:

For Y following the given Pareto distribution,

$$E(Y) = \int_\beta^\infty \alpha\beta^\alpha y^{-\alpha} dy = \alpha\beta^\alpha \left.\frac{y^{-\alpha+1}}{-\alpha+1}\right|_\beta^\infty = \alpha\beta/(\alpha-1).$$

The mean is not defined if $\alpha < 1$. Thus, a generalized MOM estimator for α cannot be expressed.

5. Problem 9.82 part b from the book (p 481) *Solution:*

The likelihood function is $L(\theta) = \theta^{-n} r^n \left(\prod_{i=1}^n y_i\right)^{r-1} \exp\left(-\sum_{i=1}^n y_i^r/\theta\right)$.

b. The log–likelihood is

$$\ln L(\theta) = -n\ln\theta + n\ln r + (r-1)\ln\left(\prod_{i=1}^n y_i\right) - \sum_{i=1}^n y_i^r/\theta.$$

By taking a derivative w.r.t. θ and equating to 0, we find $\hat\theta = \frac{1}{n}\sum_{i=1}^n Y_i^r$.

Second derivative test to confirm local maximum:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta}\ln L(\theta) &= -n/\theta + (\sum_{i=1}^n Y_i^r)/\theta^2 \\
\Rightarrow \frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\ln L(\theta) &= n/\theta^2 - 2(\sum_{i=1}^n Y_i^r)/\theta^3 \\
&= \frac{n\theta - 2(\sum_{i=1}^n Y_i^r)}{\theta^3} \\
\Rightarrow \frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\ln L(\theta)|_{\theta=(\sum_{i=1}^n Y_i^r)/n} &= \frac{\sum_{i=1}^n Y_i^r - 2(\sum_{i=1}^n Y_i^r)}{[(\sum_{i=1}^n Y_i^r)/n]^3} \\
&< 0
\end{aligned}$$

since $(\sum_{i=1}^n Y_i^r) > 0$. Since the second derivative is negative at the critical point, we see that the critical point is in fact a local maximum. Thus the MLE is $\hat\theta = (\sum_{i=1}^n Y_i^r)/n$

6. Problem 9.83 from the book (p 481)

Solution:

a. The likelihood function is $L(\theta) = (2\theta+1)^{-n}$. Let $\gamma = \gamma(\theta) = 2\theta + 1$. Then, the likelihood can be expressed as $L(\gamma) = \gamma^{-n}$. The likelihood is maximized for small values of γ. The smallest value that can safely maximize the likelihood (see Example 9.16) without violating the support is $\hat\gamma = Y_{(n)}$. Thus, by the invariance property of MLEs,

$$\hat\theta = \tfrac{1}{2}\left(Y_{(n)} - 1\right).$$

b. Since $V(Y) = \frac{(2\theta+1)^2}{12}$. By the invariance principle, the MLE is $\left(Y_{(n)}\right)^2/12$.

7. Problem 9.92 part a from the book (p 463)

Solution:

a. Following the hint, the MLE of θ is $\hat{\theta} = Y_{(n)}$.

8. Problem 9.96 from the book (p 483) *Hint:* Is there an example in section 9.7 that might be helpful?

   Solution:

   From Ex. 9.15, the MLE for σ^2 was found to be $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$. By the invariance property, the MLE for σ is $\hat{\sigma} = \sqrt{\hat{\sigma}^2} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})^2}$.

9. Problem 9.97 from the book (p 482), and (c) use the invariance property the MLE in order to find the maximum likelihood estimator for the variance of the distribution

   Solution:

   a. Since $\mu_1' = 1/p$, the MOM estimator for p is $\hat{p} = 1/m_1' = 1/\bar{Y}$.

   b. The likelihood function is $L(p) = p^n(1-p)^{\Sigma y_i - n}$ and the log–likelihood is

   $$\ln L(p) = n \ln p + (\textstyle\sum_{i=1}^{n} y_i - n)\ln(1-p).$$

   Differentiating, we have

   $$\frac{d}{dp}\ln L(p) = \frac{n}{p} - \frac{1}{1-p}(\textstyle\sum_{i=1}^{n} y_i - n).$$

   Equating this to 0 and solving for p, we obtain the MLE $\hat{p} = 1/\bar{Y}$, which is the same as the MOM estimator found in part a.

Second Derivative test to confirm local maximum:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta}\ln L(p) &= \frac{n}{p} - \frac{1}{1-p}\left(\sum_{i=1}^{n} Y_i - n\right) \\
\frac{\mathrm{d}}{\mathrm{d}\theta}\ln L(p) &= -\frac{n}{p^2} - \frac{1}{(1-p)^2}\left(\sum_{i=1}^{n} Y_i - n\right) \\
&< 0
\end{aligned}$$

since $\frac{n}{p^2} > 0$, $\frac{1}{(1-p)^2} > 0$, and $\sum_{i=1}^{n} Y_i \geq n$ (because $Y_i \geq 1$). Since the second derivative is always negative (and therefore is negative at the critical point), we see that the critical point is in fact a local maximum. Thus the MLE is $\hat{p} = \frac{1}{\bar{Y}}$.

c. Since $Y_1, \ldots, Y_n$ are iid $Geo(p)$, we know that the variance of the distribution is $\frac{1-p}{p^2}$.

By the invariance property of the MLE we have that the MLE $\widehat{\frac{1-p}{p^2}}$ for $\frac{1-p}{p^2}$ is

$$
\begin{aligned}
\frac{\widehat{1-p}}{p^2} &= \frac{1-\hat{p}}{\hat{p}^2} \\
&= \frac{1-1/\overline{Y}}{(1/\overline{Y})^2}
\end{aligned}
$$

Challenge Question:

Let $\hat{\theta}$ be the MLE for θ. Prove that the MLE for $f(\theta)$ is $f(\hat{\theta})$ when f has a unique inverse function f^{-1} such that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$ for all real x. (i.e. prove the invariance property for MLE in the case when f is one-to-one)