Homework 6

SOLUTIONS!

Book problems:

1. Problem 9.1 from the book (p 447)

   Solution:
   Refer to Ex. 8.8 where the variances of the four estimators were calculated. Thus,

   $$\text{eff}(\hat{\theta}_1, \hat{\theta}_5) = 1/3 \qquad \text{eff}(\hat{\theta}_2, \hat{\theta}_5) = 2/3 \qquad \text{eff}(\hat{\theta}_3, \hat{\theta}_5) = 3/5.$$

2. Problem 9.2 from the book (p 447)

   Solution:
   a. The three estimators a unbiased since:

   $$E(\hat{\mu}_1) = \tfrac{1}{2}\big(E(Y_1) + E(Y_2)\big) = \tfrac{1}{2}(\mu + \mu) = \mu$$

   $$E(\hat{\mu}_2) = \mu/4 + \frac{(n-2)\mu}{2(n-2)} + \mu/4 = \mu$$

   $$E(\hat{\mu}_3) = E(\overline{Y}) = \mu\,.$$

   b. The variances of the three estimators are

   $$V(\hat{\mu}_1) = \tfrac{1}{4}(\sigma^2 + \sigma^2) = \tfrac{1}{2}\sigma^2$$

   $$V(\hat{\mu}_2) = \sigma^2/16 + \frac{(n-2)\sigma^2}{4(n-2)^2} + \sigma^2/16 = \sigma^2/8 + \frac{\sigma^2}{4(n-2)}$$

   $$V(\hat{\mu}_3) = \sigma^2/n\,.$$

   Thus, $\text{eff}(\hat{\mu}_3, \hat{\mu}_2) = \dfrac{n^2}{8(n-2)}$, $\text{eff}(\hat{\mu}_3\ \hat{\mu}_1) = n/2$.

3. Problem 9.3 from the book (p 447)

   Solution:

a. $E(\hat{\theta}_1) = E(\bar{Y}) - 1/2 = \theta + 1/2 - 1/2 = \theta$. From Section 6.7, we can find the density function of $\hat{\theta}_2 = Y_{(n)}$: $g_n(y) = n(y-\theta)^{n-1}$, $\theta \le y \le \theta + 1$. From this, it is easily shown that $E(\hat{\theta}_2) = E(Y_{(n)}) - n/(n+1) = \theta$.

b. $V(\hat{\theta}_1) = V(\bar{Y}) = \sigma^2/n = 1/(12n)$. With the density in part **a**, $V(\hat{\theta}_2) = V(Y_{(n)}) = \frac{n}{(n+2)(n+1)^2}$. Thus, $\text{eff}(\hat{\theta}_1, \hat{\theta}_2) = \frac{12n^2}{(n+2)(n+1)^2}$.

4. Problem 9.6 from the book (p 447)

   Solution:

   Both estimators are unbiased and $V(\hat{\lambda}_1) = \lambda/2$ and $V(\hat{\lambda}_2) = \lambda/n$. The efficiency is $2/n$.

5. Problem 9.8 from the book (p 448)

   Solution:

   a. It is not difficult to show that $\frac{\partial^2 \ln f(y)}{\partial \mu^2} = -\frac{1}{\sigma^2}$, so $I(\mu) = \sigma^2/n$, Since $V(\bar{Y}) = \sigma^2/n$, $\bar{Y}$ is an efficient estimator of μ.

   b. Similarly, $\frac{\partial^2 \ln p(y)}{\partial \lambda^2} = -\frac{y}{\lambda^2}$ and $E(-Y/\lambda^2) = 1/\lambda$. Thus, $I(\lambda) = \lambda/n$. By Ex. 9.6, $\bar{Y}$ is an efficient estimator of λ.

6. Let $X_1, X_2, ..., X_n$ be i.i.d. $U(0, \theta)$.

   a) Let $\hat{\theta}_1 = \frac{n+1}{n} X_{(n)}$ and $\hat{\theta}_2 = 2\bar{X}$ be unbiased estimators for θ. Find $\text{eff}(\hat{\theta}_1, \hat{\theta}_2)$.
   Solution:

$$\begin{aligned} X_{(n)}/\theta &\sim Beta(n,1) \\ \Rightarrow V[\hat{\theta}_1] &= V[\frac{n+1}{n} X_{(n)}] \\ &= (\frac{(n+1)\theta}{n})^2 V[X_{(n)}/\theta] \\ &= (\frac{(n+1)\theta}{n})^2 (\frac{n}{(n+1)^2(n+2)}) \\ &= \frac{\theta^2}{n(n+2)} \end{aligned}$$

$$\begin{aligned} V[2\bar{X}] &= \frac{4}{n^2} \sum_{i=1}^{n} V[X_i] \\ &= \frac{4}{n^2} \sum_{i=1}^{n} \frac{\theta^2}{12} \\ &= \frac{4}{n^2} \frac{n\theta^2}{12} \\ &= \frac{4\theta^2}{12n} \end{aligned}$$

$$\Rightarrow \text{eff}(\hat{\theta}_1, \hat{\theta}_2) = \frac{V[\hat{\theta}_2]}{V[\hat{\theta}_1]} = \frac{\frac{4\theta^2}{12n}}{\frac{\theta^2}{n(n+2)}} = \frac{4(n+2)}{12}$$

   b) Show that $\hat{\theta}_1$ is a consistent estimator for θ

Solution:

$$\begin{aligned} V[\hat{\theta}_1] &= \frac{\theta^2}{n(n+2)} \\ \Rightarrow \lim_{n\to\infty} V[\hat{\theta}_1] &= \lim_{n\to\infty} \frac{\theta^2}{n(n+2)} \\ &= 0 \end{aligned}$$

Thus, $\hat{\theta}_1$ is consistent for θ.

7. From problem 9.1 from the book (item # 1), identify whether or not each estimator is a consistent estimator for θ or not. Include reasoning.
Solution:
Note, the variances for all of the estimators are constant, so none of them can be consistent estimators. If $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ and we had n samples from an exponential distribution instead of just 3, then the varince of $\bar{Y}$ would be $V[\bar{Y}] = \frac{\theta^2}{n^2}$ and that would be consistent because $\lim_{n\to\infty} \frac{\theta^2}{n^2} = 0$

8. From problem 9.2 from the book (item # 2), identify whether or not each estimator is a consistent estimator for μ or not. Include reasoning.
Solution:

$$\begin{aligned} \lim_{n\to\infty} V[\hat{\mu}_1] &= \frac{\sigma^2}{2} \\ \lim_{n\to\infty} V[\hat{\mu}_2] &= \frac{\sigma^2}{8} \\ \lim_{n\to\infty} V[\hat{\mu}_3] &= 0 \end{aligned}$$

Thus, only $\hat{\mu}_3$ is a consistent estimator for σ^2

9. From problem 9.3 from the book (item # 3), identify whether or not each estimator is a consistent estimator for θ or not. Include reasoning.
Solution:

$$\begin{aligned} \lim_{n\to\infty} V[\hat{\theta}_1] &= 0 \\ \lim_{n\to\infty} V[\hat{\theta}_1] &= \lim_{n\to\infty} \frac{n}{n^2+3n+3} \\ &= \lim_{n\to\infty} \frac{1}{2n+3} \leftarrow \text{L'Hôspital's Rule} \\ &= 0 \end{aligned}$$

So, both estimators are consistent for θ

10. Problem 9.26 from the book (p 457)

Solution:

a. We have that $P(\theta - \varepsilon \le Y_{(n)} \le \theta + \varepsilon) = F_{(n)}(\theta + \varepsilon) - F_{(n)}(\theta - \varepsilon)$.

- If $\varepsilon > \theta$, $F_{(n)}(\theta + \varepsilon) = 1$ and $F_{(n)}(\theta - \varepsilon) = 0$. Thus, $P(\theta - \varepsilon \le Y_{(n)} \le \theta + \varepsilon) = 1$.
- If $\varepsilon < \theta$, $F_{(n)}(\theta + \varepsilon) = 1$, $F_{(n)}(\theta - \varepsilon) = \left(\frac{\theta-\varepsilon}{\theta}\right)^n$. So, $P(\theta - \varepsilon \le Y_{(n)} \le \theta + \varepsilon) = 1 - \left(\frac{\theta-\varepsilon}{\theta}\right)^n$.

b. The result follows from $\lim_{n\to\infty} P(\theta - \varepsilon \le Y_{(n)} \le \theta + \varepsilon) = \lim_{n\to\infty} \left[1 - \left(\frac{\theta-\varepsilon}{\theta}\right)^n\right] = 1$.

11. Problem 9.29 from the book (p 458)

Solution:

$P(|Y_{(1)} - \theta| \le \varepsilon) = P(\theta - \varepsilon \le Y_{(1)} \le \theta + \varepsilon) = F_{(1)}(\theta + \varepsilon) - F_{(1)}(\theta - \varepsilon) = 1 - \left(\frac{\theta-\varepsilon}{\theta}\right)^{\alpha n}$. Since $\lim_{n\to\infty} \left(\frac{\theta-\varepsilon}{\theta}\right)^{\alpha n} = 0$ for $\varepsilon > 0$, $Y_{(1)}$ is consistent.

12. Problem 9.30 from the book (p 458)

Solution:

Note that Y is beta with $\mu = 3/4$ and $\sigma^2 = 3/5$. Thus, $E(\overline{Y}) = 3/4$ and $V(\overline{Y}) = 3/(5n)$. Thus, $V(\overline{Y}) \to 0$ and $\overline{Y}$ converges in probability to 3/4.

Challenge Question:

Let $X_1, ...X_{n_1}$ be i.i.d. $U(0, \theta)$. Show whether or not $X_{(1)} = MIN(X_1, ..., X_n)$ is a consistent estimator for θ