Homework 10

Solutions!

1. Problem 10.2 from the book (p 494)

   Solution:
   Note that Y is binomial with parameters $n = 20$ and p.
   a. If the experimenter concludes that less than 80% of insomniacs respond to the drug when actually the drug induces sleep in 80% of insomniacs, a type I error has occurred.
   b. $\alpha = P(\text{reject } H_0 \mid H_0 \text{ true}) = P(Y \le 12 \mid p = .8) = .032$ (using Appendix III).
   c. If the experimenter does not reject the hypothesis that 80% of insomniacs respond to the drug when actually the drug induces sleep in fewer than 80% of insomniacs, a type II error has occurred.
   d. $\beta(.6) = P(\text{fail to reject } H_0 \mid H_a \text{ true}) = P(Y > 12 \mid p = .6) = 1 - P(Y \le 12 \mid p = .6) = .416$.
   e. $\beta(.4) = P(\text{fail to reject } H_0 \mid H_a \text{ true}) = P(Y > 12 \mid p = .4) = .021$.

2. Problem 10.4 from the book (p 494)

   Solution:
   The parameter p = proportion of ledger sheets with errors.
   a. If it is concluded that the proportion of ledger sheets with errors is larger than .05, when actually the proportion is equal to .05, a type I error occurred.
   b. By the proposed scheme, H_0 will be rejected under the following scenarios (let E = error, N = no error):

Sheet 1	Sheet 2	Sheet 3
N	N	.
N	E	N
E	N	N
E	E	N

   With $p = .05$, $\alpha = P(NN) + P(NEN) + P(ENN) + P(EEN) = (.95)^2 + 2(.05)(.95)^2 + (.05)^2(.95) = .995125$.
   c. If it is concluded that $p = .05$, but in fact $p > .05$, a type II error occurred.
   d. $\beta(p_a) = P(\text{fail to reject } H_0 \mid H_a \text{ true}) = \text{P}(EEE, NEE, \text{ or } ENE \mid p_a) = 2p_a^2(1 - p_a) + p_a^3$.

3. Problem 10.5 from the book (p 495)

   Solution:

Under H_0, Y_1 and Y_2 are uniform on the interval (0, 1). From Example 6.3, the distribution of $U = Y_1 + Y_2$ is

$$g(u) = \begin{cases} u & 0 \le u \le 1 \\ 2-u & 1 < u \le 2 \end{cases}$$

Test 1: $P(Y_1 > .95) = .05 = \alpha$.

Test 2: $\alpha = .05 = P(U > c) = \int_c^2 (2-u)du = 2 = 2c + .5c^2$. Solving the quadratic gives the plausible solution of $c = 1.684$.

4. Problem 10.6 from the book (p 495

Solution:

The test statistic Y is binomial with $n = 36$.

a. $\alpha = P(\text{reject } H_0 \mid H_0 \text{ true}) = P(|Y - 18| \ge 4 \mid p = .5) = P(Y \le 14) + P(Y \ge 22) = .243$.

b. $\beta = P(\text{fail to reject } H_0 \mid H_a \text{ true}) = P(|Y - 18| \le 3 \mid p = .7) = P(15 \le Y \le 21 \mid p = .7) = .09155$.

5. Problem 10.7 from the book (p 495)

Solution:

a. False, H_0 is not a statement involving a random quantity.
b. False, for the same reason as part a.
c. True.
d. True.
e. False, this is given by α.
f. **i.** True.
ii. True.
iii. False, β and α behave inversely to each other.

6. Let $X_1, ..., X_{100}$ be i.i.d. $Bin(n, p)$ where n is known. Construct an appropriate large sample test statistic and corresponding Rejection region for testing $H_0 : p = p_0$ vs. $H_a : p > p_0$ at a significance level of α

Solution:

since we have over 50 randomvariables, we know that by the CLT we have,

$$\frac{\bar{X} - E[X_1]}{\sqrt{V[X_1]/100}} \sim N(0,1) \text{ (approximately)}$$

Note, that under the null we have that

$$\begin{aligned} \frac{\bar{X} - E[X_1]}{\sqrt{V[X_1]/n}} &= \frac{\bar{X} - np_0}{\sqrt{(np_0(1-p_0))/100}} \\ &= \frac{\sqrt{100}\sqrt{n}(\bar{X}/n - p_0)}{\sqrt{p_0(1-p_0)}} \end{aligned}$$

$$\Rightarrow \frac{\sqrt{100}\sqrt{n}(\bar{X}/n - p_0)}{\sqrt{p_0(1-p_0)}} \quad \sim \quad N(0,1) \text{ (approximately)}$$

Since $H_a : p > p_0$, we know that our rejection region will be of the form $RR = \{x : x > k\}$ where k is chosen such that our significance equals α. Fixing $P(\text{type I error}) = \alpha$ we have

$$\begin{aligned}
\alpha &= P(\text{type I error}) \\
&= P(\bar{X} \in RR|H_0) \leftarrow \text{We start with } \bar{X} \text{ because we are constructing a large sample test} \\
&= P(\bar{X} > k|H_0) \\
&= P(\frac{\bar{X} - np_0}{\sqrt{(np_0(1-p_0))/100}} > \frac{k - np_0}{\sqrt{(np_0(1-p_0))/100}}|p = p_0) \\
&\approx P(Z > \frac{k - np_0}{\sqrt{(np_0(1-p_0))/100}}) \text{ Where } Z \sim N(0,1)
\end{aligned}$$

So, this gives us that $\frac{k-np_0}{\sqrt{(np_0(1-p_0))/100}} = Z_\alpha$, the $1-\alpha$ percentile of $N(0,1)$. Solving for k we get $k = Z_\alpha\sqrt{(np_0(1-p_0))/100} + np_0$. So depending on the statistic we use for our test, the following rejection regions produce the equaivalent test:

Statistic	RR
$\frac{\sum_{i=1}^{100} X_i/100 - np_0}{\sqrt{(np_0(1-p_0))/100}}$	$\{x : x \geq Z_\alpha\}$
$\bar{X}$	$\{x : x \geq Z_\alpha\sqrt{(np_0(1-p_0))/100} + np_0\}$
$\sum_{i=1}^{100} X_i$	$\{x : x \geq 100Z_\alpha\sqrt{(np_0(1-p_0))/100} + 100np_0\}$

7. Problem 10.19 from the book (p 504)

   Solution:
   H_0: $\mu = 130$, H_a: $\mu < 130$. Using the large sample test for a mean, $z = \frac{128.6-130}{2.1/\sqrt{40}} = -4.22$ and with $-z_{.05} = -1.645$, H_0 is rejected: there is evidence that the mean output voltage is less than 130.

8. Problem 10.37 from the book (p 510)

   Solution:
   With H_0: $\mu = 130$, this is rejected if $z = \frac{\bar{y}-130}{\sigma/\sqrt{n}} < -1.645$, or if $\bar{y} < 130 - \frac{1.645\sigma}{\sqrt{n}} = 129.45$. If $\mu = 128$, then $\beta = P(\bar{Y} > 129.45 \mid \mu = 128) = P(Z > \frac{129.45-128}{2.1/\sqrt{40}}) = P(Z > 4.37) = .0000317$.

9. Problem 10.43 from the book (p 510) *Note:* for 10.43, assume that the CLT applies when the sample size is larger than 30 (In class I said 50, but here 30 is fine)
   Solution:

Let μ_1 and μ_2 denote the mean dexterity scores for those students who did and did not (respectively) participate in sports.

a. For H_0: $\mu_1 - \mu_2 = 0$ vs. H_a: $\mu_1 - \mu_2 > 0$ with $\alpha = .05$, the rejection region is $\{z > 1.645\}$ and the computed test statistic is

$$z = \frac{32.19 - 31.68}{\sqrt{\frac{(4.34)^2}{37} + \frac{(4.56)^2}{37}}} = .49 .$$

Thus H_0 is not rejected: there is insufficient evidence to indicate the mean dexterity score for students participating in sports is larger.

b. The rejection region, written in terms of the sample means, is

$$\overline{Y}_1 - \overline{Y}_2 > 1.645\sqrt{\frac{(4.34)^2}{37} + \frac{(4.56)^2}{37}} = 1.702 .$$

Then, $\beta = P(\overline{Y}_1 - \overline{Y}_2 \le 1.702 \mid \mu_1 - \mu_2 = 3) = P\left(Z \le \frac{1.703-3}{\sigma_{\bar{Y}_1 - \bar{Y}_2}}\right) = P(Z < -1.25) = .1056.$

10. You work for Circuts'R'Us, which among other things, manufactures electrical circuts. The output voltage on the circuts that you inspect is supposed to be 130, but recently you think that there may have been a glitch in the manufacturing process causing the voltages to be lower than they should be. You sample 64 independent circuts off of the assembly line, and you got a sample mean of 128.5 (where each circut's volatge is beleived to be normally distributed with a mean of μ and a standard deviation of 4). Using this information, you would like to conduct a hypothesis test at a significance level of α to decide whether the circuts being produced have a voltage lower than what they are supposed to be.

a) Give the statement of the Hypotheses, an appropriate test statistic based on $\bar{X}$, the sample mean of the voltage measurements, and corresponding rejection region

Solution:
Let μ be the mean voltage of all circuts off of the assembly line Hypothesis Statements:
$H_0 : \mu = 130$ Vs. $H_a : \mu < 130$
Test Statistic:
$\frac{\bar{X}-130}{4/\sqrt{64}}$,which, under the null hypothesis is approximately distributed $N(0, 1)$
Rejection Region:
$RR = \{x : x < -z_\alpha\}$, where Z_α is the $1 - \alpha$ percentile of the $N(0, 1)$ distribution

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Equivalent statistic and rejection region 1: $\bar{X}$ and $RR = \{x : x < 130 - 4/\sqrt{64}Z_\alpha\}$, where Z_α is the $1 - \alpha$ percentile of the $N(0, 1)$ distribution

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Equivalent statistic and rejection region 2: $\bar{X}$ and $RR = \{x : x < Z^*_{1-\alpha}\}$, where $Z^*_{1-\alpha}$ is the α percentile of the $N(130, 4^2/64)$ distribution

b) Conduct the test at the $\alpha = .025$ significance level. Be sure to clearly state

your conclusion (i.e. reject, fail to reject). *Hint:* $Z_{.025} = 1.96$
Solution:
Our test statistic is

$$\begin{aligned}\frac{\bar{X} - 130}{4/\sqrt{64}} &= \frac{-1.5}{4/8} \\ &= -3\end{aligned}$$

Since our test statistic is -3 which is less than $-Z_{.025} = -1.96$, we see that our statistic falls into the rejection region, and therefore we reject the null hypothesis in favor of the alternative hypothesis

Challenge Question:

Let, X, Y be i.i.d. $N(\mu, 1)$. Imagine a hypothesis test of $H_0 : \mu = \mu_0$ vs. $H_a : \mu > \mu_0$, where we conduct our test in two stages. First we observe $X = x$. If $x - \mu_0 > Z_c$ (where Z_c is the $1 - c$ percentile of $N(0, 1)$), then we reject. If we do not reject in this first stage, then we move on to the second stage where we observe $Y = y$. If $y - \mu_0 > Z_c$ then we reject. Find c such that α, the overall significance of the test, is 0.05